Public Safety under Imperfect Taxation

Nicolas Treich

Toulouse School of Economics, INRA, University Toulouse Capitole

Yuting Yang

Toulouse School of Economics, University Toulouse Capitole

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Abstract

In this paper, we examine theoretically the effect of the imperfection of the taxation system on the optimal level of public safety provision. We compare three taxation systems: first-best, income and uniform taxation. Under wealth heterogeneity, there is normally more public safety under first-best than under uniform taxation, but there can be more or less public safety under first-best than under income taxation. Under risk heterogeneity, the comparison depends on whether the heterogeneity concerns the baseline risk or the reduction in risk. Specifically, under heterogeneous baseline risk, public safety under first-best is lower, and not greater, than under either income or uniform taxation when the utility has constant relative risk aversion. Under heterogeneity in risk reduction, public safety under first-best is in general greater than under uniform or income taxation. We also consider distortionary taxation, namely income taxation under endogenous labor supply. We show that there can be more or less public safety under first-best compared to distortionary taxation. Overall, we conclude that the imperfections of the taxation system cannot generically justify more or less public safety provision. This implies that there is no fundamental reason to always adjust downwards the value of statistical life (VSL) because of imperfect taxation, nor to systematically assume a marginal cost of public funds larger than one for the benefit cost analysis of public safety projects.

Keywords: Safety provision, imperfect taxation, distortionary taxation, wealth heterogeneity, risk aversion, value of statistical life.

1 Introduction

Mortality reduction represents a significant part of the benefit of many public projects. It has been estimated to account for more than 90% of the monetized benefit of the Clean Air Act (U.S.EPA, 2011). Usually, the standard approach to compute the benefits of safety is to multiply the estimated number of lives saved by the average value of a statistical life (VSL) in the affected population. It is well known however that this common approach in benefit cost analysis relies on the assumption that the taxation system is perfect. In this paper, we relax this assumption, and we examine how the imperfection of the taxation system affects the optimal level of public safety.

The vast majority of the literature on public safety has ignored the issue of imperfect taxation. In academia, the literature has examined both theoretically and empirically how VSL varies with the characteristics of individuals or of the decision making environment (Viscusi and Aldy, 2003; Andersson and Treich, 2011). But to the best of our knowledge, this literature has not thoroughly studied the specific effect of imperfect taxation. In the practice of benefit cost analysis, imperfect taxation is often accounted by introducing a marginal cost of public funds larger than one. This practice implies that the cost of the project should be augmented due to imperfect taxation, and in turn that public safety should be reduced. This practice seems problematic because public economics theory is inconclusive in general about whether distributional objectives may lead to a greater or lower level of public expenditures in second best rather than in first best environments (Gaube, 2000).

Accounting for imperfect taxation in the evaluation of public safety projects is important for several reasons. First, it is well documented that the taxation system is imperfect in both developed and developing countries, and that the degree of imperfection vary widely across the world. Second, from a policy perspective, various guidelines encourage safety regulators to also include "distributive impacts," "equity," or "environmental justice" in benefit cost analysis. But it is also well known that concrete methodologies for evaluating such additional impacts remain undeveloped (Adler, 2008). Third, the literature in public economics has long debated the issue of optimal provision of public good when distribution matters, and it seems natural to examine a specific but important domain of application like safety provision. Indeed, safety usually raises strong equity issues that call for a careful and systematic analysis of distributive incidence. Therefore, it seems important to better understand conceptually the limitation of standard benefit cost analysis under imperfect taxation, and to get more insights about the "biases" induced by this method when the assumption of perfect taxation is relaxed. Accordingly, we believe that a natural starting point in order to get more insights is to develop a comparative statics analysis of the degree of imperfection of the taxation system on the optimal level of public safety.

In our analysis, we proceed as follows. We compare three types of taxation systems: first-best, income and uniform taxation. We consider in turn two types of heterogeneity, wealth and mortality risk. Our main results are the following. Under wealth heterogeneity, we show that there is normally more public safety under first-best than under uniform taxation, but that there can be more or less public safety under first-best than under income taxation. Under risk heterogeneity, we show that the comparison depends on whether the heterogeneity concerns the baseline risk or the reduction in risk. Specifically, under heterogeneous baseline risk, public safety is lower than under either income or uniform taxation when the utility has constant relative risk aversion. Under heterogeneity in risk reduction, public safety is in general greater than under uniform or income taxation. We finally consider distortionary taxation, namely the case where income tax affects labor supply. We show that there can be more or less public safety under first-best compared to distortionary taxation depending on whether the labor cost is "tangible" (i.e. commensurable with wealth) or not.

From this analysis, we conclude that the imperfection of the taxation system cannot generically justify more or less public safety provision. In particular, it indicates that there is no a priori reason to adjust downwards the VSL because the expenditures in safety should be financed with imperfect or distortionary taxation. Similarly, there is no a priori reason to systematically assume a marginal cost of public funds larger than one for the benefit cost analysis of safety projects. Instead, it suggests that these adjustments should be made on a case by case basis and in particular depend on the type of imperfections of the taxation system as well as on the sources of heterogeneity and risk preferences.

Our paper builds on an extensive literature related to VSL. It is well established theoretically that VSL represents the marginal rate of substitution between wealth and mortality risk.

Thus, VSL is sensitive to the wealth and baseline mortality risk, defined as the "high-payment effect" and "dead-anyway effect" (Pratt and Zeckhauser, 1996). Following this basic framework, researchers have extended the model by incorporating other characteristics of risk preferences, such as risk aversion (Eeckhoudt and Hammitt, 2004) and background risk (Eeckhoudt and Hammitt, 2001), to analyze their effect on VSL. Hammitt and Robinson (2011) also looks at adjusting VSL with country level estimated income elasticities to transfer one country's VSL to another. Kaplow (2005) states the discrepancy between theoretical an empirical estimates income elasticity of VSL. But none of these papers consider the case of imperfect taxation. Armantier and Treich (2004) derive an approximation and a numerical analysis of the bias induced by using average VSL under uniform taxation. We add to the literature by considering the effect of imperfect taxation on public safety and VSL.

Our analysis is also related to the literature of the optimal provision of public good under imperfect taxation. The most fundamental theory is a conjecture put forward by Pigou (1928), which states with distortionary tax, the cost inflicted on consumers exceeds the direct production cost of providing the public good. Thus, at the optimum, marginal benefit of the public good should be greater than the marginal cost, which results in the under provision of public good. This argument is further developed into the marginal cost of public fund (MCPF), which was then incorporated in to the Samuelson's rule.[1] If Pigou's conjecture holds, the value of MCPF should be greater than 1. However, conflicting views have been raised in regards to Pigou's conjecture. Stiglitz and Dasgupta (1971), Atkinson and Stern (1974), Ballard and Fullerton (1992) investigate the conditions under which Pigou's conjecture doesn't hold (MCPF<1). They claim that MCPF greatly depends on the effect of the public good on labor supply and other taxed activities. Gaube (2000) contributes to the discussion by integrating redistributive concerns with heterogeneous agents and concludes that equity consideration may increase public expenditure in the second best. We bridge the two literatures by applying the theory of optimal public good provision to the specific question of public safety provision and develop further implications on the effect of imperfect taxation on MCPF.

[1]Samuelson (1954) provides a basic theory for public good provision: the necessary condition to reach Pareto Optimality of public good provision is to equalize the sum of marginal rates of substitution (MRS) between a public and a private good and the marginal rate of transformation (MRT) $\sum MRS = MRT$. The modified Samuelson's rule has $\sum MRS = MCPF \times MRT$. For more on MCPF, see Atkinson and Stern (1974), Ballard and Fullerton (1992).

The remainder of this paper is organized as follows. Section 2 presents the model. Section 3 examines the effect of imperfect taxation on public safety under wealth heterogeneity, while. section 4 considers mortality risk heterogeneity. Section 5 analyzes the impact of a distortionary tax on labor supply and safety provision. Section 6 discusses the link of our analysis with VSL. Section 7 discusses some policy implications. Section 8 concludes. The appendix contains additional derivations and proofs of our results.

2 The Model

2.1 The Framework

We consider model with H agents. Each agent has an expected utility,

$$E(U_i) = p_i u_i(w_i), \forall i \in 1, ..., H \tag{2.1}$$

where p_i denotes the probability of survival, w_i denotes his exogenous wealth and $u_i(w_i)$ is the utility he gets if he survives. For simplicity, we assume that the utility function is the same for all agents ($u_i = u$) and that the bequest motive is normalized to zero. [2]

Suppose a utilitarian government is choosing the level of safety provision, e.g. how much to invest in improving air quality. The financing of public safety G comes from individual taxation t_i ($G = \sum_{i=1}^{H} t_i$). The probability of survival is assumed to be an increasing function of G. In order to determine the socially optimal level of public safety, the government solves the following welfare maximization problem:

$$\begin{aligned} \max_{\{t_i\}_{i\in\{1,...H\}}} \quad & \sum_{i=1}^{H} \Big(p_i(G) u(w_i - t_i) \Big) \\ \text{s.t.} \quad & G \le \sum_{i=1}^{H} t_i \end{aligned} \tag{2.2}$$

Setting the Lagrangian:

$$\mathscr{L} = \sum_{i=1}^{H} \Big(p_i(G) u(w_i - t_i) \Big) + \mu(G - \sum_{i=1}^{H} t_i) \tag{2.3}$$

[2] The possibility of a bequest motive is discussed in the appendix.

the first order conditions (focs) with respect to G and t_i give

$$p_i(G^*)u'(w_i - t_i^*) = \mu, \quad \forall i \tag{2.4}$$

$$\sum_{i=1}^{H} p_i'(G^*)u(w_i - t_i^*) = \mu \tag{2.5}$$

The focs indicate that a utilitarian planner would equalize the after tax expected marginal utility of wealth across individuals.

Replacing μ in equation 2.5, we get

$$\sum_{i=1}^{H} \frac{p_i'(G^*)u(w_i - t_i^*)}{p_i(G^*)u'(w_i - t_i^*)} = 1 \tag{2.6}$$

Equation 2.6 characterizes the efficient condition to achieve the optimal level of public safety provision. It corresponds to the Samuelson's rule of equalizing social marginal benefit to the social marginal cost of providing public safety.

To achieve this first-best level of public safety provision, individual lump-sum tax is used. We denote this tax system as "perfect taxation", as there is perfect redistribution without any distortion. However, in practice, perfect taxation is difficult to implement due to informational and political constraints. Therefore, in this paper, we refer to any other taxation schemes that deviate from the first-best as "imperfect taxation". The purpose of this paper is to find out how the level of public safety differs from the first-best under imperfect taxation.

2.2 Technologies

To carry out the analyses, we first make some assumptions on the form of the utility and survival function.

Regarding the utility function $u(w)$, we assume it is strictly positive, increasing and concave: $u > 0, u' > 0, u'' < 0$, and thrice differentiable. The utility is assumed to be positive because bequest is set at zero, and surviving is strictly preferred to dying. We will use to two common utility forms CRRA and Exponential utility to illustrate the analyses. With CRRA utility, $u(w) = \frac{w^{1-\gamma}}{1-\gamma}, \gamma \in (0,1)$; with Exponential utility, $u(w) = \frac{1-e^{-\alpha w}}{\alpha}, \alpha > 0$. Two coefficient that are analytically important is the coefficient of relative risk aversion $R(w) = -w\frac{u''(w)}{u'(w)}$ and the coefficient of fear of ruin $FR(w) = \frac{u(w)}{u'(w)}$ (Foncel and Treich, 2005). The only class of utility function that has linear fear of ruin is CRRA. CRRA also has the property that $R(w) = \gamma$.

For the survival function, we assume it is positive, increasing, linear or concave: $p_i(\cdot) > 0$, $p_i(\cdot)' > 0$, $p_i(\cdot)'' \leq 0$, and $p_i(G) < 1$ for all is. For the simulations, we will use the functional from $p(G) = a + b\frac{G}{1+G}$, $0 \leq a < 1 - b$ and $b \geq 0$.[3]

2.3 The Benchmark

In the following, we study the effect of heterogeneity by considering a model with two agents who differ either in wealth or morality risk.[4] Therefore, the benchmark (first-best) case under the assumptions made above is the following:

$$\max_{t_1, t_2} p_1(t_1 + t_2)u(w_1 - t_1) + p_2(t_1 + t_2)u(w_2 - t_2) \tag{2.7}$$

the focs are

$$\begin{aligned} p_1'(t_1^* + t_2^*)u(w_1 - t_1^*) + p_2'(t_1^* + t_2^*)u(w_2 - t_2^*) &= p_1(t_1^* + t_2^*)u'(w_1 - t_1^*) \\ &= p_2(t_1^* + t_2^*)u'(w_2 - t_2^*) \end{aligned} \tag{2.8}$$

The focs imply that the first best optimal tax level would equalize individual's after tax expected marginal utility of wealth. Intuitively, an individual with higher initial wealth level and/or lower probability of survival should be subject to a higher tax rate.

In the following sections, we compare the optimal level of public safety under perfect and imperfect taxation. Heterogeneities in wealth and risk are considered separately to understand the effect of each type of individual heterogeneity. The imperfect tax systems considered are a uniform lump-sum tax for all agents and an income tax with a uniform tax rate.

[3]When the specific functions are used, we normally have $G > 0$.
[4]The generalization to multiple agents is discussed in the appendix.

3 Wealth Heterogeneity

First, we consider the case of heterogeneous wealth w_1 and w_2 ($w_1 > w_2$). The first best is characterized by individual taxation (t_i). Deviating from the first best, we have a case of uniform tax (t_U) and a case of an income tax (τ). Thus, the utilitarian social planner's solves the following maximization problems under the three tax regimes.

First-best:

$$\max_{t_1,t_2} p(t_1+t_2)[u(w_1-t_1)+u(w_2-t_2)] \tag{3.1}$$

Uniform Tax:

$$\max_{t_U} p(2t_U)[u(w_1-t_U)+u(w_2-t_U)] \tag{3.2}$$

Income Tax:

$$\max_{\tau} p\big(\tau(w_1+w_2)\big)[u(w_1(1-\tau))+u(w_2(1-\tau))] \tag{3.3}$$

Taking the focs with respect to the decision variables, we get:

First-best:

$$\frac{p(G_F)}{p'(G_F)} = 2\frac{u(w_1-t_1^*)}{u'(w_1-t_1^*)} = 2\frac{u(w_2-t_2^*)}{u'(w_2-t_2^*)} \tag{3.4}$$

Uniform Tax:

$$\frac{p(G_U)}{p'(G_U)} = 2\frac{u(w_1-t_U^*)+u(w_2-t_U^*)}{u'(w_1-t_U^*)+u'(w_2-t_U^*)} \tag{3.5}$$

Income Tax:

$$\frac{p(G_I)}{p'(G_I)} = (w_1+w_2)\frac{u(w_1(1-\tau^*))+u(w_2(1-\tau^*))}{w_1u'(w_1(1-\tau^*))+w_2u'(w_2(1-\tau^*))} \tag{3.6}$$

where $G_F = t_1^* + t_2^*$, $G_U = 2t_U^*$ and $G_I = \tau^*(w_1+w_2)$.[5]

We are interested in the ranking of G_F, G_U and G_I. The comparison between first-best and uniform tax and first-best and income tax are carried out separately in the remainder of this section.

[5] Given the assumptions made, the second-order conditions hold globally. The Hessian of $U(t_1^*, t_2^*)$ is a negative definite symmetric matrix.

3.1 First Best and Uniform Tax Comparison

Proposition 1. *Under wealth heterogeneity, with $u'''(x) \geq 0$, the optimal level of public safety in the first-best is higher than that with uniform taxation ($G_F > G_U$).*

Proof. The first order conditions can be rewritten as

$$\frac{p(G_F)}{p'(G_F)} = 2\frac{u(w_1 - t_1^*) + u(w_2 - t_2^*)}{u'(w_1 - t_1^*) + u'(w_2 - t_2^*)} \tag{3.7}$$

$$\frac{p(G_U)}{p'(G_U)} = 2\frac{u(w_1 - t_U^*) + u(w_2 - t_U^*)}{u'(w_1 - t_U^*) + u'(w_2 - t_U^*)} \tag{3.8}$$

We demonstrate the result by contradiction. Assume $\frac{t_1^*+t_2^*}{2} \leq t_U^*$. Then

$$\begin{aligned} u(w_1 - t_U^*) + u(w_2 - t_U^*) &\leq u(w_1 - \frac{t_1^* + t_2^*}{2}) + u(w_2 - \frac{t_1^* + t_2^*}{2}) && \text{by } u' > 0 \\ &< u(w_1 - t_1^*) + u(w_2 - t_2^*) && \text{since } w_1 - t_1^* = w_2 - t_2^* \text{ and } u'' < 0 \end{aligned}$$

Similarly, we have

$$\begin{aligned} u'(w_1 - t_U^*) + u'(w_2 - t_U^*) &\geq u'(w_1 - \frac{t_1^* + t_2^*}{2}) + u'(w_2 - \frac{t_1^* + t_2^*}{2}) && \text{by } u'' < 0 \\ &\geq u'(w_1 - t_1^*) + u'(w_2 - t_2^*) && \text{since } w_1 - t_1^* = w_2 - t_2^* \text{ and } u''' \geq 0 \end{aligned}$$

Therefore, under $u''' \geq 0$, we have

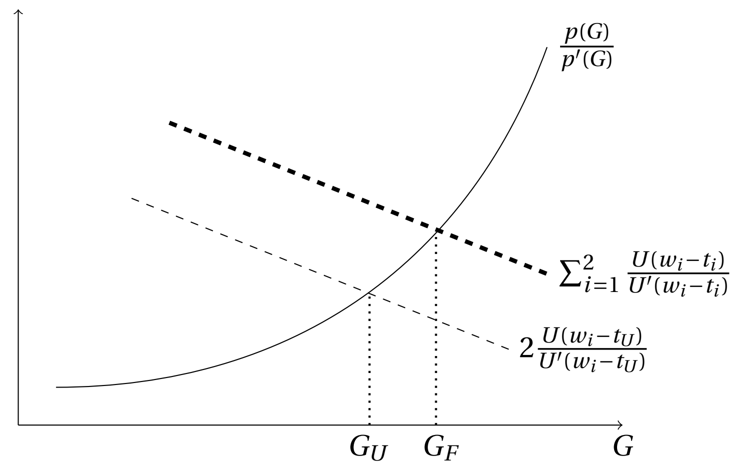
$$\frac{p(G_U)}{p'(G_U)} = 2\frac{u(w_1 - t_U^*) + u(w_2 - t_U^*)}{u'(w_1 - t_U^*) + u'(w_2 - t_U^*)} < 2\frac{u(w_1 - t_1^*) + u(w_2 - t_2^*)}{u'(w_1 - t_1^*) + u'(w_2 - t_2^*)} = \frac{p(G_F)}{p'(G_F)}$$

Since $\frac{p(G)}{p'(G)}$ is an increasing function in G, this implies $\frac{t_1^*+t_2^*}{2} > t_U^*$, a contradiction. □

Proposition 1 shows that, for utility function with its third derivative (prudence) non-negative, the optimal level of public safety is higher in the first best than with a uniform tax (see figure 3.1). The assumption $u''' \geq 0$, or prudence (Kimball, 1990) is common in the literature. It is a necessary condition for decreasing absolute risk aversion. The intuition of the result lies in the fact that tax as an policy instrument have two purposes: redistribution and financing. In the first best, since there are two possibly different taxes available, both purposes could be taken care of. Whereas with a uniform tax, the financing objective could be obtained, but redistribution could not.

[To be completed...]

Figure 3.1: Illustration of comparison between first best and uniform tax



To illustrate the result with a specific example, consider two individuals, one with wealth of 1000 and one with 10. They both have the same CRRA utility $u(w) = \frac{w^{0.5}}{0.5}$ and survival function $p(G) = \frac{G}{1+G}$. Under perfect taxation, individual 1 pays 516.7 and individual 2 receives a subsidy of 473.3. Therefore, the total investment on public safety is 43.5. However, when the taxation is limited to a uniform tax, then each individual pays 7.1. The investment on public safety is significantly lower in this case.

3.2 First Best and Income Tax Comparison

Remark 1. *The optimal level of public safety in the first-best could be above, below or equal to the level with income tax.*

Table 3.1 presents some simulations on the optimal public safety provision under three specific cases. We look at two different utility forms in particular, CRRA and exponential utility. With CRRA utility, the optimal level is the same under first-best and income tax. With exponential utility, when the degree of risk preference (parameter a in the utility function) varies, the level of provision may be higher or lower in first best than with income tax.

Table 3.1: Simulations of first-best and income tax

		Utility		
		CRRA	Exponential	
		$\gamma = 0.5$	$a = 0.02$	$a = 0.001$
Tax Rate	t_1	276.64	580.727	272.87
	t_2	-223.354	80.727	-227.13
	τ	0.0355	0.329	0.034
Welfare	First Best	105.599	99.826	1101.3
	Income Tax	104.079	99.736	983.15
		$G_F = G_I$	$G_F > G_I$	$G_F < G_I$

Note: Simulation in Mathematica. Taking $p(x) = \frac{x}{1+x}$, $w_1 = 1000$, $w_2 = 500$, CRRA utility $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and exponential utility $u(x) = \frac{1-e^{-ax}}{a}$.

Table 3.1 indicates that the optimal level of public safety depends on the specific class of utility function and its parameters in each class. In the following, we further study the case of CRRA utility and check whether this result of equal level of public safety in first-best and income tax always holds.

Remember that CRRA utility has the form $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$, $\gamma < 1$. Therefore, $u'(w) = w^{-\gamma}$ and the fear of ruin $\frac{u(w)}{u'(w)} = \frac{w}{1-\gamma}$ is linear in w.

Substituting the utility function into equation 3.4 and 3.6, we get

First-Best:

$$\frac{p(G_F)}{p'(G_F)} = \frac{w_1 - t_1^*}{1-\gamma} + \frac{w_2 - t_2^*}{1-\gamma} = \frac{w_1 + w_2 - G_F}{1-\gamma} \tag{3.9}$$

Income Tax:

$$\frac{p(G_I)}{p'(G_I)} = (w_1 + w_2)\frac{1-\tau^*}{1-\gamma} = \frac{w_1 + w_2 - G_I}{1-\gamma} \tag{3.10}$$

Note that the equality holds when $\tau^*(w_1 + w_2) = t_1^* + t_2^*$.

Proposition 2. *Under wealth heterogeneity, if the utility function satisfies CRRA, then the optimal level of public safety in the first-best is the same as that with income taxation ($G_F = G_I$).*

Proposition 2 suggests that although income tax cannot perfectly redistribute wealth compare to the first-best, if the utility form satisfies a particular form, the optimal level of safety could still be the same. This special property of CRRA utility ($\frac{u}{u'}$ linear in w) is instrumental to this equalizing result. Notice that with CRRA utility, the RHS of equation 3.6 is equivalent to the expression

$$\frac{u((w_1+w_2)(1-\tau^*))}{u'((w_1+w_2)(1-\tau^*))}.$$

This indicates that under CRRA the optimal income tax does not care about the distribution of wealth among the two agent, but only about the sum of wealth. Therefore, as long as total wealth is the same, the optimal level of public safety would always coincide in the first-best and income tax, regardless of how the wealth is distributed.

However, in the case of uniform tax, the distribution of wealth matters for the optimal safety level. If we express equation 3.5 as

$$\frac{p(G_U)}{p'(G_U)} = 2\alpha\frac{u(w_1-t_U^*)}{u'(w_1-t_U^*)} + 2\beta\frac{u(w_2-t_U^*)}{u'(w_2-t_U^*)} \tag{3.11}$$

where $\alpha \equiv \frac{1}{1+\frac{u'(w_2-t_U^*)}{u'(w_1-t_U^*)}}$, $\beta \equiv \frac{1}{1+\frac{u'(w_1-t_U^*)}{u'(w_2-t_U^*)}}$, $\alpha+\beta=1$.

Since by construction, $u'(w_2-t_U^*) > u'(w_1-t_U^*)$. Therefore, $0<\alpha<0.5$, $0.5<\beta<1$. Denote $\eta \equiv \frac{w_1-w_2}{w_1+w_2}$ as the degree of wealth heterogeneity, η takes value between 0 and 1, with $\eta=0$ meaning perfect equality and $\eta=1$ perfect inequality. It is obvious that α decreases in η and β increases in η (when $\eta=0$, $\alpha=\beta=0.5$). Therefore, unless wealth is equally distributed, the optimal level of public safety in uniform tax can never be equal to that in the first-best. In the following section, we will analyze how the optimal level in uniform tax changes with wealth heterogeneity.

3.3 More Wealth Inequality

Remark 2. *Wealth inequality does not affect the first-best and income tax (if CRRA) optimal level of public safety, but reduces the level in uniform tax and income tax (if not CRRA).*

Proof. Set $\frac{w_1}{W}+\frac{w_2}{W}=1$, $\frac{w_1}{W}-\frac{w_2}{W}=\eta$, where $W=w_1+w_2$. η denotes the level of wealth inequality.

Express the RHS of equation 3.4 in terms of η,

$$F(\eta) = 2\frac{u(\frac{W-G_F}{2})}{u'(\frac{W-G_F}{2})} \tag{3.12}$$

we find that G_F is independent from η.

Rewrite the RHS of equation 3.5 as a function of η:

$$F(\eta) = 2\frac{u\Big((1+\eta)\frac{W}{2} - t_U^*\Big) + u\Big((1-\eta)\frac{W}{2} - t_U^*\Big)}{u'\Big((1+\eta)\frac{W}{2} - t_U^*\Big) + u'\Big((1-\eta)\frac{W}{2} - t_U^*\Big)} \tag{3.13}$$

Taking the first order derivative of function $F(\eta)$ with respect to η, we get that

$$\frac{dF(\eta)}{d\eta} = 2\frac{\frac{W}{2}[(u_1' - u_2')(u_1' + u_2') - (u_1 + u_2)(u_1'' - u_2'')]}{(u_1' + u_2')^2} \tag{3.14}$$

where $u_1 = u\Big((1+\eta)\frac{W}{2} - t_U^*\Big)$ and $u_2 = u\Big((1-\eta)\frac{W}{2} - t_U^*\Big)$. By assumption, $\eta > 0$, therefore, $u_1' < u_2'$, $u_1'' > u_2''$. Thus, $\frac{dF(\eta)}{d\eta} < 0$. As $F(\eta)$ is a decreasing function of η, we must have that as with a fixed total wealth, as η increases, the optimality condition gives a lower G_U.

Rewrite the RHS of equation 3.6 as a function of η: :

$$F(\eta) = W\frac{u\Big((1+\eta)\frac{W}{2}(1-\tau^*)\Big) + u\Big((1-\eta)\frac{W}{2}(1-\tau^*)\Big)}{(1+\eta)\frac{W}{2}u'\Big((1+\eta)\frac{W}{2}(1-\tau^*)\Big) + (1-\eta)\frac{W}{2}u'\Big((1-\eta)\frac{W}{2}(1-\tau^*)\Big)} \tag{3.15}$$

$$\frac{dF(\eta)}{d\eta} = \frac{\frac{W}{2}(1-\tau^*)\Big[(u_1' - u_2')(\frac{1+\eta}{2}u_1' + \frac{1-\eta}{2}u_2') - (u_1 + u_2)((1+\eta)u_1'' - (1-\eta)u_2'')\Big]}{\Big((1+\eta)\frac{W}{2}u_1' + (1-\eta)\frac{W}{2}u_2'\Big)^2} \tag{3.16}$$

where $u_1 = u\Big((1+\eta)\frac{W}{2}(1-\tau^*)\Big)$ and $u_2 = u\Big((1-\eta)\frac{W}{2}(1-\tau^*)\Big)$, again we have $\frac{dF(\eta)}{d\eta} < 0$. This implies that the optimal level of public safety under income taxation decreases with increasing wealth inequality, when utility is not CRRA.[6] □

To illustrate this remark with an extreme example: consider two agents with the same wealth ($\eta = 0$), the optimality condition in uniform tax would coincide with the first-best $G_F = G_U$. However, if one agent possesses the entire wealth ($\eta = 1$), we already know that the optimal level in the first-best remains the same (through redistribution from the wealthy to the poor), yet with uniform tax, the poor can no longer be taxed, which would result in zero level of public safety provision.

[6]When utility is CRRA, $F(\eta) = \frac{W(1-\tau^*)}{1-\gamma}$, $\frac{dF(\eta)}{d\eta} = 0$.

3.4 Welfare Analysis

We have observed that for CRRA utility, public safety level under income tax and first best are the same. Does that imply using income tax is as good as using individual lump-sum tax? Welfare analysis has to be carried out to reach a conclusion.

Remark 3. *Under wealth heterogeneity, for any utility function, social welfare under income tax is always higher than that under uniform tax, but lower than the first best.*

Proof. In the first best, it is always possible to choose $t_1 = \tau w_1$ and $t_2 = \tau w_2$, so that welfare under first best can do at least as well as under income tax.

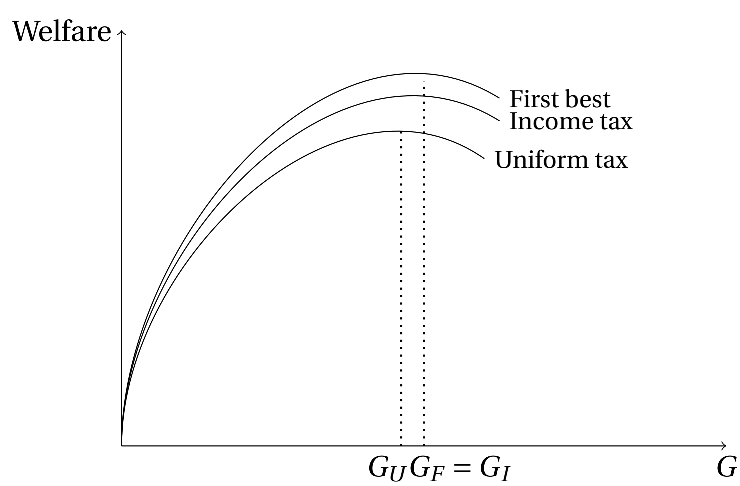
Between income and uniform tax, we have for every given level of G such that $G = \tau(w_1 + w_2) = 2t_U$,

$$p(G)(u(w_1 - \tau w_1) + u(w_2 - \tau w_2)) > p(G)(u(w_1 - t_U) + u(w_2 - t_U)),$$

by the concavity of the utility function. Thus, the welfare with income tax is always greater than that with uniform tax. □

It follows that in the first best, planner equalizes the after tax wealth of the agents. Since income tax is closer to the first best in terms of redistribution, it must be that income tax result in a higher welfare level than uniform tax.

Figure 3.2: Illustration of the welfare level under different tax systems



4 Mortality Risk Heterogeneity

In this section, we consider individual heterogeneity in mortality risk, $p_1(G) > p_2(G)$. For example, individuals may be exposed to the risk differently before the implementation of a mortality risk reduction project, which means that their baseline risk is different. Similarly, individuals could have the same baseline risk, but some may benefit more from the project than others. This indicates different levels of risk reduction. Intuitively, perfect taxation would be taxing more the agent that is more vulnerable (lower p_i) to the risk due to the dead-anyway effect (Pratt and Zeckhauser, 1996). However, under imperfect taxation, the optimal level of public safety may be different from the first-best.

The planner's problems can be written as follows.

First-Best:

$$\max_{t_1,t_2} p_1(t_1+t_2)u(w-t_1)+p_2(t_1+t_2)u(w-t_2) \tag{4.1}$$

Uniform Tax:

$$\max_{t_U} \Big(p_1(2t_U)+p_2(2t_U)\Big)u(w-t_U) \tag{4.2}$$

Income Tax:

$$\max_{\tau} \Big(p_1(2\tau w)+p_2(2\tau w)\Big)u(w(1-\tau)) \tag{4.3}$$

Note that income tax is equivalent to uniform tax in this scenario as there is no heterogeneity in wealth. Indeed, we can always set $\tau = \frac{t_U}{w}$ to have $w(1-\tau) = w - t_U$ and obtain $G_I = G_U$. Therefore, we can just focus our analysis on the uniform tax case.

Taking the first order conditions, we get:

First-Best:

$$\begin{aligned} p_1'(G_F)u(w-t_1^*)+p_2'(G_F)u(w-t_2^*) &= p_1(G_F)u'(w-t_1^*) \\ &= p_2(G_F)u'(w-t_2^*) \end{aligned} \tag{4.4}$$

Uniform Tax:

$$\frac{p_1(G_U)+p_2(G_U)}{p_1'(G_U)+p_2'(G_U)} = \frac{2u(w-t_U^*)}{u'(w-t_U^*)} \tag{4.5}$$

where $G_F = t_1^* + t_2^*$ and $G_U = 2t_U^*$. The first best focs also implies $t_1 < t_2$.

Again, we are interested in the comparison between G_F and G_U. To examine that, we rearrange equations 4.4 and 4.5 as follows:

$$
\begin{aligned}
\frac{p_1(G_F)+p_2(G_F)}{p_1'(G_F)+p_2'(G_F)} =& \frac{u(w-t_1^*)}{u'(w-t_1^*)}+\frac{u(w-t_2^*)}{u'(w-t_2^*)} \\
&+\underbrace{\frac{p_2(G_F)}{p_1'(G_F)+p_2'(G_F)}}_{A}\underbrace{\frac{u(w-t_1^*)-u(w-t_2^*)}{u'(w-t_1^*)}}_{B}\underbrace{\left(\frac{p_1'(G_F)}{p_1(G_F)}-\frac{p_2'(G_F)}{p_2(G_F)}\right)}_{C}
\end{aligned}
\tag{4.6}
$$

$$
\frac{p_1(G_U)+p_2(G_U)}{p_1'(G_U)+p_2'(G_U)}=\frac{2u(w-t_U^*)}{u'(w-t_U^*)} \tag{4.7}
$$

When utility is CRRA, $\frac{u}{u'}$ is linear, and $G_F = G_U$ if the last part of equation 4.6 is zero. Given the assumptions we made on the functional forms, we know $A > 0$ and $B > 0$. So it seems that when there is only risk heterogeneity, the expression that may drive the comparison between G_F and G_U is $\frac{p_1'(G_F)}{p_1(G_F)} - \frac{p_2'(G_F)}{p_2(G_F)}$. In the case $\frac{p_1'(G_F)}{p_1(G_F)} > \frac{p_2'(G_F)}{p_2(G_F)}$, $C > 0$, and the first-best optimal level must be higher than with uniform tax ($G_F > G_U$). Conversely, if $\frac{p_1'(G_F)}{p_1(G_F)} < \frac{p_2'(G_F)}{p_2(G_F)}$, $C < 0$, then first-best optimal level would be lower than with uniform tax ($G_F < G_U$).

As mentioned in the example above, heterogeneity in risk may be conceived in two distinct ways: baseline risk and risk reduction. To further understand the different effects, we separately analyze heterogeneous baseline risk and heterogeneous risk reduction.

4.1 Heterogeneous Baseline Risk

With heterogeneous baseline risk, agents have different baseline survival probability p_i, but receive the same level of benefit from the public safety project $\varepsilon(G)$. The survival function could be expressed as

$$
p_i(G) = p_i + \varepsilon(G).
$$

Here following the assumptions we made about the survival function, $\varepsilon(\dot{)}$ is positive, increasing and weakly concave. Assuming $p_1 > p_2$, then

$$
\frac{p_1'(G_F)}{p_1(G_F)} = \frac{\varepsilon'(G_F)}{p_1+\varepsilon(G_F)} < \frac{\varepsilon'(G_F)}{p_2+\varepsilon(G_F)} = \frac{p_2'(G_F)}{p_2(G_F)}
$$

. Thus $C < 0$ and $G_F < G_U$. [7]

[7] For the second order conditions, see Appendix A.2.

Proposition 3. *Under heterogeneous baseline risk, if utility is CRRA, optimal level of public safety is lower in the first-best than with uniform or income tax ($G_F < G_U = G_I$).*

[[To be completed...]

This reverse result with CRRA utility comparing with the heterogeneity in wealth is very intriguing. With heterogeneous baseline risk, perfect taxation aims to equate the weighted after tax marginal utility of wealth of each individual. This weight is denoted by the survival probability. In other words, individual will higher survival probability should be taxed less. This is consistent with the dead-anyway effect (Pratt and Zeckhauser, 1996) which indicates the willingness to pay of individuals increase in their mortality risk. When the only available taxation tool is a uniform tax, to get closer to the first-best objective, as the after tax marginal utility of wealth is the same for both individuals (without wealth heterogeneity), the only way to narrow the gap of the weighted marginal utility is through increasing the risk reduction $\varepsilon(G)$. The higher G, the smaller the difference between $p_1 + \varepsilon(G)$ and $p_2 + \varepsilon(G)$. Therefore, the government may need to provide even more safety than the first-best optimum. With a uniform tax, the dead-anyway effect is exacerbated on the aggregate level.

Moreover, the structure of the VSL objective function implies that we not only care about the marginal increase in probability of survival given the technology, but also the baseline value. This is unique comparing to other separable objective functions used commonly in the public economics literature. This is why the inverse result occurs.]

Consider an extreme case, agent 1's exposure to the risk is almost zero ex-ante ($p_1 \rightarrow 1$) and agent 2 is very exposed ($p_2 \rightarrow 0$). In the first best, government will be taxing a very small amount from agent 1 and a large amount from agent 2 ($t_2 >> t_1$). With a uniform tax, the result indicates that the government is willing to sacrifice the consumption utility from agent 1 in order to save the life of agent 2. Because the heterogeneity in baseline risk is the only driving force of the result, the larger the difference, the bigger G_U is compared to G_F.

If we relax the constraint $\frac{u}{u'}$ linear, the result would be ambiguous (see Table 4.1).

Table 4.1 shows that depending on the utility form, optimal provision of safety could be lower or higher with uniform tax than in the first best.

Table 4.1: Simulation for heterogeneous baseline risk

		Utility	
		CRRA	Exponential
		$\gamma = 0.5$	$a = 0.02$
Tax Rate	t_1	-71.205	667.821
	t_2	342.999	679.207
	t	136.936	673.232
Welfare	First Best	97.402	88.083
	Income Tax	96.690	88.082
		$G_F < G_U$	$G_F > G_U$

Note: Simulation in Mathematica. Taking $p(G) = p_i + \frac{\alpha G}{1+2\alpha G}$, $p_1 = 0.5$, $p_2 = 0.3$, $\alpha = 0.01$. CRRA utility $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$. Exponential utility $U(w) = \frac{1-e^{-aw}}{a}$, with $a = 0.02$, initial wealth $w = 1000$.

4.2 Heterogeneous risk reduction

Now we look at the case where agents have the same baseline risk but different level of risk reduction.

$$p_i(G) = p + \delta_i(G)$$

We consider the two cases:

1. $\delta_i(G) = \delta_i G$

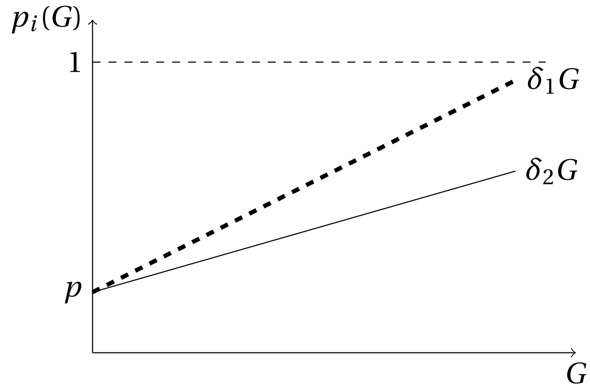
   Proposition 4. *Under heterogeneous linear risk reduction and $p_i(G) = p + \delta_i G$, if utility is CRRA, then the optimal level of safety provision in the first-best is higher than that with uniform or income tax ($G_F > G_U = G_I$).*

   Proof. Assume $\delta_1 > \delta_2$, then

$$\frac{p_1'(G_F)}{p_1(G_F)} = \frac{\delta_1}{p + \delta_1 G_F} > \frac{\delta_2}{p + \delta_2 G_F} = \frac{p_2'(G_F)}{p_2(G_F)}$$

   Thus $C > 0$, $G_F > G_U$. □

Figure 4.1: Illustration of the heterogeneous risk reduction case $p_i = p + \delta_i G$



2. $\delta_i(G)$ non-linear

   When risk reduction is non-linear (see figure 4.2)

$$p_i(G) = p + \delta_i(G)$$

   Assume $\delta_i(\cdot) > 0$, $\delta_i'(\cdot) > 0$, $\delta_i''(\cdot) \leq 0$ for all is and $\delta_1(G) > \delta_2(G)$, then

$$\frac{p_1'(G_F)}{p_1(G_F)} = \frac{\delta_1'(G_F)}{p + \delta_1(G_F)}$$

$$\frac{p_2'(G_F)}{p_2(G_F)} = \frac{\delta_2'(G_F)}{p + \delta_2(G_F)}$$

   As the first order derivative of the probability function has a crossing point, when $G \leq \hat{G}$, $C < 0$. When $G > \hat{G}$, C could be positive, zero or negative, which makes the relationship between G_F and G_U ambiguous.

[To be completed...

The intuition for the linear case is clear. Perfect taxations impose lower tax rate on the agent that is more responsive to the safety project to achieve efficiency. Under imperfect taxation, providing lower safety could diminish the difference between $\delta_1 G$ and $\delta_2 G$ (figure 4.1 shows the gap between risk reduction is smaller when G is smaller). Therefore, to ensure efficiency, government needs to set G_U to be lower than G_F.

In the case of non-linear risk reduction, the situation becomes more complicated. For a

Figure 4.2: Illustration of the heterogeneous risk reduction case $p_i = p + \delta_i(G)$

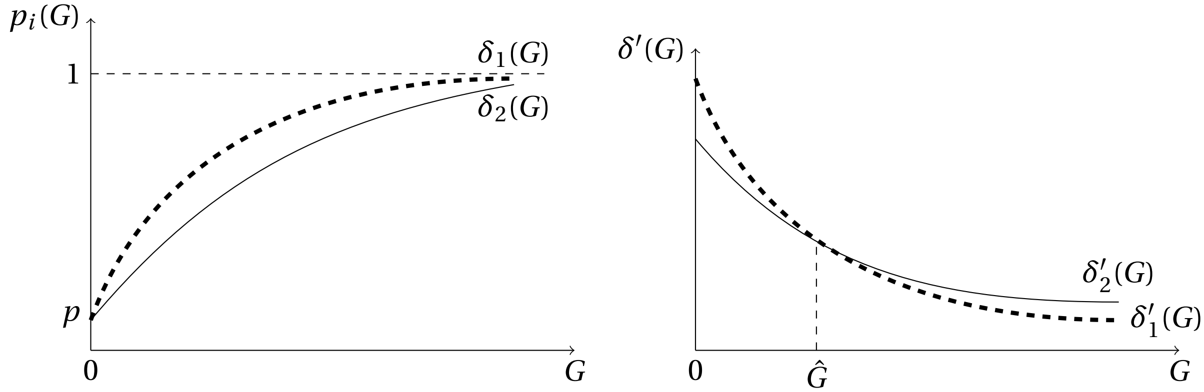


Table 4.2: Public Safety under different utility form and survival function

	CRRA		Exponential	
	Linear p	Non-linear p	Linear p	Non-linear p
G_F	908.033	395.896	1658.69	1404.28
G_U	888.889	398.235	1656.92	1400.91
	$G_F > G_U$	$G_F < G_U$	$G_F > G_U$	$G_F > G_U$

Note: Simulation in Mathematica. Taking non-linear $p(G) = 0.1 + \frac{\alpha_i G}{1+2\alpha_i G}$, $\alpha_1 = 0.01$, $\alpha_2 = 0.0025$, linear $p(G) = p + \delta_i G$, $\delta_1 = 0.0001$, $\delta_2 = 0.00005$. CRRA utility $U(w) = \frac{w^{1-\gamma}}{1-\gamma}$, $\gamma = 0.5$, exponential utility $U(w) = \frac{1-e^{-aw}}{a}$, $a = 0.2$, initial wealth $w = 1000$.

concave risk reduction, the marginal rate of reduction must cross at some level of G (figure 4.2 right graph). As the left graph in figure 4.2 shows, the gap between risk reduction first increase and then decrease as G increases. Therefore, to ensure efficiency (minimize difference of weighted marginal utility of wealth), G_U could be higher or lower than G_F.]

4.3 More Risk Inequality

[To be completed...]

Remark 4. *Baseline risk inequality has no effect on the optimal level of public safety in the uniform tax, but reduces the level in the first-best.*

Proof. Set $p_1 = \bar{p} + \eta + \varepsilon(G)$ and $p_2 = \bar{p} - \eta + \varepsilon(G)$. η denotes the degree of inequality of baseline risk.

The foc of uniform tax is independent from η, therefore G_U remains constant as η changes.

The LHS of the foc of first best is independent from η, but the RHS is not. Thus, we express part of equation 4.4 $(A * C)$ as a function of η:

$$F(\eta) = \frac{1}{2}\Big(\frac{\bar{p} - \eta + \varepsilon(G_F)}{\bar{p} + \eta + \varepsilon(G_F)} - 1\Big) \tag{4.8}$$

Taking the first order derivate of function $F(\eta)$ with respect tp η, we get

$$\frac{dF(\eta)}{d\eta} = -\frac{\bar{p} + \varepsilon(G_F)}{(\bar{p} + \eta + \varepsilon(G_F))^2} < 0 \tag{4.9}$$

Thus, the RHS of equation 4.4 decreases in η, which implies G_F decreases in η. □

Remark 5. *Risk reduction inequality has no effect on the optimal level of public safety in the uniform tax, but,*

1. *with linear risk reduction, increases the level in the first-best.*
2. *with non-linear risk reduction [To be complete...]*

Proof. Set $p_1 = p + (\bar{\delta} + \eta)G$ and $p_2 = p + (\bar{\delta} - \eta)G$. η denotes the degree of heterogeneity of risk reduction.

Follow the same steps of the previous proof, the foc of uniform tax is independent from η, therefore G_U remains constant as η changes.

The LHS of the foc of first best is independent from η, but the RHS is not. Thus, we express part of equation 4.4 $(A * C)$ as a function of η:

$$F(\eta) = \frac{1}{2\bar{\delta}}\Big((\bar{\delta} + \eta)\frac{p + (\bar{\delta} - \eta)G_F}{p + (\bar{\delta} + \eta)G_F} - (\bar{\delta} - \eta)\Big) \tag{4.10}$$

$$\frac{dF(\eta)}{d\eta} = \frac{p(p + \bar{\delta}G_F)}{\bar{\delta}(p + (\bar{\delta} + \eta)G_F)^2} > 0 \tag{4.11}$$

Thus, the RHS of equation 4.4 increases in η, which implies G_F increases in η. □

5 Distortionary Taxation

This section focuses on the use of distortionary taxation with identical agents, such that there is no heterogeneity and distributive concerns. Consider an economy with H identical individuals. Each individual maximizes his utility by choosing the level of consumption (c) and labor supply(l). The utility function is $u(c,l)$ $(u_c > 0, u_{cc} < 0, u_l < 0$, and $u_{ll} < 0)$. $w \in \mathbb{R}_{++}$ denotes the exogenous unit wage rate. Individuals are assumed to be small so they do not take into account the feedback effect of taxation. First-best is characterized by lump-sum taxation. Here, second-best is optimality under income tax.

In the first best, each agent takes the lump-sum tax (g) as given and maximizes his utility subject to the budget constraint.

Individual's problem:

$$\begin{aligned} \max_{c_g, l_g} \quad & p(G)u(c_g, l_g) \\ \text{s.t.} \quad & c_g = wl_g - g \end{aligned} \tag{5.1}$$

The optimal individual's decision is characterized by $c_g^*(g), l_g^*(g) : u_c w + u_l = 0$. u_c and u_l denote the first order derivative of the utility function with respect to its first and second argument.

The government then determines the optimal level of taxation subject to the revenue requirement for public safety to provision.

Planner's problem:

$$\begin{aligned} \max_{g} \quad & Hp(G)u\Big(c_g^*(g), l_g^*(g)\Big) \\ \text{s.t.} \quad & G = Hg \end{aligned} \tag{5.2}$$

Foc:

$$\frac{p(G)}{Hp'(G)} = \frac{u\Big(c_g^*(g), l_g^*(g)\Big)}{u_c\Big(c_g^*(g), l_g^*(g)\Big)} \tag{5.3}$$

In the second best, the government imposes a distortive income tax, in the sense that the tax discourages labor effort. Each agent maximizes his utility taking τ as given:

$$\begin{aligned} \max_{c_t, l_t} \quad & p(G)u(c_t, l_t) \\ \text{s.t.} \quad & c_t = w l_t (1-\tau) \end{aligned} \tag{5.4}$$

The optimal individual's decision is characterized by $c_t^*(\tau), l_t^*(\tau) : u_c w(1-\tau) + u_l = 0$.

Planner's problem:

$$\begin{aligned} \max_{\tau} \quad & H p(G_t) u\Big(c_t^*(\tau), l_t^*(\tau)\Big) \\ \text{s.t.} \quad & G_t = H w l_t^*(\tau^*)\tau^* \end{aligned} \tag{5.5}$$

Foc:

$$\frac{p(G_t)}{Hp'(G_t)} = \frac{u\Big(c_t^*(\tau), l_t^*(\tau^*)\Big)}{u_c\Big(c_t^*(\tau), l_t^*(\tau^*)\Big)}(1+\varepsilon_{l\tau^*}) \tag{5.6}$$

where $\varepsilon_{l\tau^*} = \frac{\partial l}{\partial \tau} / \frac{l}{\tau}$ denotes the labor supply elasticity of income tax.[8]

Comparing equation 5.3 and 5.6, the only additional term is $1+\varepsilon_{l\tau^*}$, which could be interpreted as the distortionary effect of income tax. Depending on the value of this labor supply elasticity, the first best public safety provision (G) could be greater or lower than that with a income tax (G_t).

The value of $\varepsilon_{l\tau}$ depends on the properties of the utility function. Here we consider two cases where the labor effort can be tangible (i.e. commensurable with wealth) $u(w,l) = u(wl - e(l))$ or non-tangible $u(w,l) = u(wl) - e(l)$ (with $e(l) > 0$, $e'(l) > 0$, $e''(l) > 0$).

Proposition 5. *Under distortionary tax with identical agents,*

1. *if labor cost is tangible, then labor supply decreases with increasing tax rate ($\varepsilon_{l\tau^*} < 0$) and the optimal level of public safety is lower under distortionary tax than in the first-best ($G > G_t$).*

2. *if labor cost is non-tangible, the elasticity of labor supply is positive (negative) ($\varepsilon_{l\tau^*} > 0 (< 0)$) if the level of relative risk aversion is greater (lower) than 1 ($R > 1 (< 1)$), and the optimal*

[8]Second order conditions for both the individual problem and planner's problem are satisfied globally.

level of public safety is respectively lower (higher) under distortionary tax than in the first-best ($G > (<) G_t$).

Proof. As $\varepsilon_{l\tau^*} = \frac{\partial l}{\partial \tau} / \frac{l}{\tau}$ and $\frac{l_t^*(\tau^*)}{\tau^*} > 0$, to obtain the sign of $\varepsilon_{l\tau^*}$, we just need the sign of $\frac{\partial l^*}{\partial \tau}$. For $\frac{\partial l^*}{\partial \tau}$, we can obtain its value by applying the implicit function theorem (IFT).[9]

1. Tangible labor cost $u(w,l) = u(wl - g - e(l))$

$$\frac{\partial l^*(\tau)}{\partial \tau} = -\frac{\frac{\partial^2 U}{\partial l \partial \tau}}{\frac{\partial^2 U}{\partial^2 l}} = -\frac{w}{e''(l^*)} \tag{5.7}$$

   By assumption, $e''(l) > 0$, then $\frac{\partial l^*}{\partial \tau} < 0 \implies \varepsilon_{l\tau^*} < 0 \implies G > G_t$.

2. Non-tangible labor cost $u(w,l) = u(wl - g) - e(l)$

$$\frac{\partial l^*(\tau)}{\partial \tau} = \frac{u'' w^2 l^* (1-\tau) + u' w}{u'' w l^* (1-\tau)^2 - e''(l^*)} \tag{5.8}$$

   By assumption, the denominator is negative. If we denote the coefficient of relative risk aversion as $R(c) = -c\frac{u''(c)}{u'(c)}$, in this case $R = -wl(1-\tau)\frac{u''}{u'}$. If $R < 1$, then the numerator in equation 5.8 >0 $\implies \varepsilon_{l\tau^*} < 0 \implies G > G_t$. If $R \geq 1$, then the numerator $\leq 0 \implies \varepsilon_{l\tau^*} \geq 0$ $\implies G \leq G_t$.

□

If the utility function satisfies the CRRA form, $u = \frac{w^{1-\gamma}}{1-\gamma}$, then $R = \gamma$. As we have restricted utility to be positive, here $\gamma \leq 1$. Thus, with a non-tangible cost of labor, under CRRA and positive utility assumption, we still have $G > G_t$. However, with other utility functions, it may occur that $G \leq G_t$. For example, for exponential utility of the form $u(w) = \frac{1-e^{-\alpha w}}{\alpha}$, we have $R = \alpha w$, which could be greater than one under our assumptions.

How can we explain ths ambiguous result under non-tangible cost of effort? It follows that taxation creates both a substitution effect and an income effect which works in opposite directions in determining optimal labor supply. If tax rate increases, implicitly the consumption

[9] Implicit function theorem: $\forall (x,y) \in V \; f(x,y) = 0 \Leftrightarrow \phi(x) = y$.The derivative of ϕ at point x_0 ($\phi(x_0, y_0) = 0$) can be expressed as $\frac{\partial \phi}{\partial x} x_0 = -\frac{\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)}$.

good is more expensive compared with leisure. Thus the substitution effect would reduce labor supply. However, with higher tax rate, wealth decreases and the marginal utility of wealth increases. Income effect would increase labor supply to avoid being poor. When substitution effect dominates the income effect, the rise in tax rate would reduce overall labor supply and vice-versa. Essentially, with $R > 1$, the curvature of the utility function is large. Thus, a small decrease in wealth would imply a large increase in the marginal utility. This would result in a larger income effect than the substitution effect, which induce higher labor supply with higher tax rate. When labor effort is "tangible", the curvature of the utility function does not play a role in optimal decision making ($e'(l) = w(1-\tau)$). Whereas when labor effort is separable from the utility of wealth, the curvature of utility function effectively decides labor supply ($e'(l) = w(1-\tau)u'(w(1-\tau)l)$. Thus, only in the "non-tangible" case, the ambiguous result occurs.

We have therefore illustrated that optimal level of public safety with distortionary tax may be above or below the first best level, depending on the degree of distortion. The degree of distortion is determined on the labor supply elasticity of income tax $\varepsilon_{l\tau^*}$.

6 Link with VSL

[To be completed...]

In practice, it is common that policy making agencies that implement safety projects, e.g. the U.S. Environmental Protection Agency (EPA) and the U.S. Department of Transportation (DOT), use the Value of a Statistical Life (VSL) as the measure of a major component of a safety benefit. They often use a single value of VSL obtained from meta-analysis for all projects. This VSL is equivalent to average individual VSLs.

To understand how VSL fits into our analysis, we can organize equation 2.6 in the general model as

$$\sum_{i=1}^{H} p_i'(G^*) VSL_i = 1 \tag{6.1}$$

where VSL_i is the value of statistical life corresponding to each individual i:

$$VSL_i \equiv \frac{u(w_i - t_i^*)}{p_i(G^*)u'(w_i - t_i^*)} \tag{6.2}$$

The VSL represents the marginal rate of substitution between wealth and mortality risk. Observe that if $p_i'(G)$ is independent from VSL_i, then equation 6.1 is equivalent to

$$\frac{1}{n}\sum_{i=1}^{H} VSL_i = \frac{1}{p_i'(G^*)}, \tag{6.3}$$

which equates the average VSL to the marginal cost of saving a life. Equation 6.2 also indicates that taxation need to be "perfect" to make average VSL the proper measure of benefit.

From the analysis we can infer that using average VSL is only justified under perfect taxation and independent risk reductions. If the taxation system is imperfect, the average VSL may create an over- or underestimation of project benefit. Assuming only imperfect tax is available, what would be the direction of bias if we continue to use average VSL?

6.1 Wealth Heterogeneity

We have already established in proposition 1 that optimal level of public safety under uniform tax is lower than that in the first best (under certain conditions). How can this translate to average VSL?

Observing the first order conditions, we could see that equation 3.5 can be expressed as

$$p'(G_U)(\sum_{i=1}^{2} VSL_i - A) = 1, \tag{6.4}$$

where $VSL_i = \frac{u(w_i - t_U^*)}{p(G_U)u'(w_i - t_U^*)}$, $A = \frac{(u_2' - u_1')(u_1 u_2' - u_2 u_1')}{p(G_U)(u_1' + u_2')u_1' u_2'} > 0$ ($u_i' = u'(w_i - t_U)$) by $u' > 0$ and $\frac{u}{u'}$ increasing.

It it obvious that under uniform taxation, the use of average VSL is not supported. Average VSL would result in an over provision of public good than the optimal level under uniform taxation.

Similarly, equation 3.6 can also be expressed in terms of VSL:

$$p'(G_I)(\sum_{i=1}^{2} VSL_i - B) = 1, \tag{6.5}$$

where $VSL_i = \frac{u(w_i(1-\tau^*))}{p(G_U)u'(w_i(1-\tau^*))}$, $B = \frac{(w_2 u_1 u_2' - w_1 u_2 u_1')(u_2' - u_1')}{u_1' u_2'(w_1 u_1' + w_2 u_2')}$. The sign of B depends on the sign of $w_2 u_1 u_2' - w_1 u_2 u_1'$, which is ambiguous depending on the utility function. With CRRA utility, $B = 0$, equation 6.5 is reduce to $p'(G_I)\sum_{i=1}^{2} VSL_i = 1$, which supports average VSL. Whereas with other utility functions, we could not determine whether the use of average VSL is supported or not.

6.2 Risk Heterogeneity

7 Policy Implications

8 Concluding Remarks

In this paper, we have examined theoretically the effect of imperfect taxation system on the optimal level of public safety provision. Comparative statistics analysis has been carried out between three taxation systems under wealth or risk heterogeneity.

The following table summarizes the results obtained in this paper:

Table 8.1: Summary of Results

	Heterogeneous Wealth		Heterogeneous Baseline Risk		Heterogeneous Risk Reduction				Distortionary Tax	
$p(G)$					Linear		Non-linear			
$U(w)$	CRRA	$U''' > 0$	CRRA	Others	CRRA	$\frac{1-e^{-aw}}{a}$ $a > 0$	CRRA	$\frac{1-e^{-aw}}{a}$ $a > 0$	$\varepsilon_{l\tau} < 0$	$\varepsilon_{l\tau} \geq 0$
G_F ? G_U	$>$	$>$	$<$	$>,=,<$	$>$	$>$	$<$	$=$	-	-
G_F ? G_I	$=$	$>,=,<$	$<$	$>,=,<$	$>$	$>$	$<$	$=$	-	-
G_I ? G_U	$>$	$>,=,<$	$=$	$=$	$=$	$=$	$=$	$=$	-	-
G_F ? G_t	-	-	-	-	-	-	-	-	$>$	$\leq$

From the results we can infer that the imperfections of the taxation system cannot generically justify more or less public safety provision. This implies that there is no fundamental reason to always adjust downwards the value of statistical life (VSL) because of imperfect taxation, nor to systematically assume a marginal cost of public funds larger than one for the benefit cost analysis of public safety projects. To determine the optimal level, it appears to understand the source of heterogeneity in the population and the underlying utility function.

[To be completed...]

A Extentions

A.1 Bequest Motive

With a bequest motive, the general maximization problem in equation 2.2 can be revised as follows:

$$\begin{aligned} \max_{\{t_i\}_{i\in\{1,...H\}}} \quad & \sum_{i=1}^{H} \Big(p_i(G)u(w_i - t_i) + (1 - p_i(G)v(w_i - t_i) \Big) \\ \text{s.t.} \quad & G \leq \sum_{i=1}^{H} t_i \end{aligned} \tag{A.1}$$

where u and v are concave state dependent utilities of consumption in the states where the individual lives or dies respectively. As is common in the literature, we could consider the case that $v = ku$ for some $k \in [0, 1)$. This means that the utility if you die is proportionally lower than the utility if you survive. Now we demonstrate all results of the paper carry out with this particular case.

The first-best maximization problem can be expressed as

$$max_{t_1,t_2}\pi_1(G)u(w_1 - t_1) + \pi_2(G)u(w_2 - t_2) \tag{A.2}$$

where $\pi_i(G) = k + (1-k)p_i(G)$, $\pi(\cdot) > 0$, $\pi'(\cdot) > 0$, $\pi''(\cdot) \leq 0$. It is straight forward that G_F, G_U and G_I keep the same ordering, restoring the original result.

More general form of bequest motive is left for future research.

A.2 Second Order Conditions

In order to have the second order condition satisfied for the maximization problem, we must have the Hessian of equation 4.4 to be negative definite. This would require that $M_{11} < 0$ $(\frac{\partial^2 L}{\partial t_1^2} < 0)$ and $M_{22} > 0$ $(\frac{\partial^2 L}{\partial t_1^2}\frac{\partial^2 L}{\partial t_2^2} - (\frac{\partial^2 L}{\partial t_1 \partial t_2})^2 > 0)$.

The first condition is easy to show. For the second condition, denote:

$$A_1 = p_1''u_1, A_2 = p_2''u_2, B_1 = p_1'u_1', B_2 = p_2'u_2', C_1 = p_1u_1'', C_2 = p_2u_2''$$

If

$$(A_1 + A_2)(C_1 + C_2) - (B_1 - B_2)^2 - 2B_1C_2 - 2B_2C_1 + C_1C_2 > 0,$$

then the second order condition is satisfied globally.

[To be completed...]

A.3 Multiple Agents

[To be completed...]

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