ON THE PROPERTIES OF NON-MONETARY MEASURES FOR RISKS†

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Abstract: This paper investigates how welfare losses for facing risks change as the risk environment of the decision-maker is altered. To that aim, we define the risk apportionment of order n (RA-n) utility premium as a measure of pain associated with facing the passage from one risk to a riskier one. Changes in risks are expressed through the concept of stochastic dominance of order n. Three configurations of risk exposures are considered. The paper first shows how the RA-n utility premium is modified when initial wealth becomes riskier. Second, the paper provides conditions on individual preferences for superadditivity of the RA-n utility premium. Third, the paper investigates welfare changes of merging increases in risks. These results offer new interpretations of the sign of higher derivatives of the utility function.

Keywords: risk apportionment, superadditivity, RA-n utility premium

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1. INTRODUCTION

The issue of how the presence of multiple risks modify individual behaviour in the face of another risk has been leading to a prolific literature during the last decades.[1] Most of these studies use monetary measures to analyze behaviour towards risk, the most well-known being the risk premium and the willingness to pay. More recently, a few papers have used non-monetary measures to provide new behavioural results in the face of risks. In particular, the concept of utility premium originally introduced by Friedman and Savage (1948) has regained interest. For instance, Eeckhoudt and Schlesinger (2006) rely on the utility premium to propose a unified approach to explain the meaning of the signs of the successive derivatives of the utility function. Eeckhoudt and Schlesinger (2009) also reexamine the properties of the utility premium and explain the relevance of this tool for decision making. Recently, Crainich and Eeckhoudt (2008) and Courbage and Rey (2010) used non-monetary measures of prudence and temperance to extract behavioural results. Such non-monetary measures not only offer alternative tools to analyse the individual loss of welfare due to the presence of risks, but also allow for much simpler conditions on individual preferences to predict behaviour towards risks.

An issue of importance when dealing with measures of risks is how these measures react to a riskier environment. In particular, knowing how welfare losses of facing increases in risks change as a function of the number of risk exposures offers crucial knowledge on how individuals react to riskier environment. To address these issues, this paper defines the risk apportionment of order n (RA-n) utility premium as a measure of pain associated with facing the passage from one risk to a worse one. Changes in risks are expressed through the specific concept of stochastic dominance of order n.

The paper first shows how the RA-n utility premium is modified when individual wealth becomes riskier. It makes it possible to generalise earlier results on the topic, and in particular those of Courbage and Rey (2010). Second, the paper provides conditions on individual preferences for superadditivity and subadditivity of the RA-n utility premium. A measure is said to be superadditive or convex in the number of risks if the measured value of two risks is superior to the sum of the values of each risk; the opposite holding for subadditivity. Superadditivity/subadditivity sheds light on whether risks are self-aggravating for individuals. For instance, Eeckhoudt and Gollier (2001) show that risk vulnerability (see Gollier and Pratt, 1996) is a sufficient condition for superadditivity of the risk premium. The concept of subadditivity of risks measures has also been popularised through the definition of coherent risk measures defined by Artzner et al. (1999).

[1] See Eeckhoudt and Gollier (2013) for a review.

Third, the paper uses the RA-n utility premium to investigate a related but different issue, i.e. whether welfare loss changes when increases in risks are merged instead of facing them separately. This makes it possible to extend the recent results of Ebert et al. (2017) on mutual aggravation of risk changes in terms of mutual improvements of risk changes.

The paper shows that the degree of pain due to facing an increased risk grows when the initial wealth becomes riskier if the signs of the successive derivatives of the utility function alternate in signs, i.e. when preferences exhibit mixed risk-aversion as defined by Caballé and Pomansky (1996). It also shows that mixed risk aversion of order 4 is sufficient for superadditivity of the RA-n utility premium. Finally, mixed risk-aversion is also shown to drive welfare changes of merging increases in risks. The results of this paper therefore provide new interpretations of the sign of higher derivatives of the utility function.

The paper is organised as follows. Section 2 introduces the benchmark model for non-monetary measures of risk and in particular the RA-n utility premium. Section 3 investigates the effect of riskier initial wealth on the RA-n utility premium. Section 4 addresses the conditions on individuals preferences for superadditivity/subadditivity of non-monetary measures of risks. Section 5 deals with welfare changes of merging increases in risks and provides a link with the work of Ebert et al. (2017). Section 6 finally offers a short conclusion.

2. THE BENCHMARK MODEL

2.1. Non-monetary measures in the face of risks

Non-monetary measures in the face of risks stem from the work of Friedman and Savage (1948) who used expected utility theory to define risk aversion and introduced two ways for its measure. The two measures reflect the subjective cost of risk for a risk averter.

Let an individual's final wealth be represented by $x + \tilde{\epsilon}$ where x ($x > 0$) denotes the initial wealth of the individual and $\tilde{\epsilon}$ is a zero-mean random variable[2]. The first measure of risk aversion in the face of the risk $\tilde{\epsilon}$ at wealth level x is a monetary measure, *the risk premium* $\pi(x, \tilde{\epsilon})$, and is such that:

$$E[u(x + \widetilde{\epsilon})] = u(x - \pi(x, \tilde{\epsilon})), \tag{1}$$

where u denotes the individual's von Neumann-Morgenstern utility function (with $u'(x) \geq 0\ \forall x$) and E denotes the expectation operator. $\pi(x, \tilde{\epsilon})$ is the amount of money that the agent is ready to pay to get rid of the zero-mean risk $\widetilde{\epsilon}$. $\pi(x, \tilde{\epsilon}) \geq 0$ if and only if the individual is risk-averse ($u''(x) \leq 0\ \forall x$).

[2]We assume that the support of $\tilde{\epsilon}$ is defined such that $x + \epsilon$ is in the domain of u.

The second one is a non-monetary measure of risk aversion, *the utility premium*, $w_A(x, \tilde{\epsilon})$:

$$w_A(x, \widetilde{\varepsilon}) = u(x) - E[u(x + \widetilde{\varepsilon})]. \quad (2)$$

$w_A(x, \widetilde{\varepsilon})$ measures the degree of "pain" associated with facing the risk $\widetilde{\varepsilon}$, where pain is measured by the loss in expected utility from adding the risk $\widetilde{\varepsilon}$ to wealth x. From Jensen's inequality, $w_A(x) \geq 0$ if and only if $u''(x) \leq 0 \; \forall x$[3].

Prudence is known as preference for a zero-mean risk in the wealthier state of nature. The prudence utility premium as introduced by Crainich and Eeckhoudt (2008), denoted $w_P(x, \widetilde{\varepsilon})$, measures the increase in pain of facing the risk $\widetilde{\varepsilon}$ in the presence of a sure loss $l > 0$. This is defined as follows:

$$w_P(x, \widetilde{\varepsilon}) = u(x - l) - E[u(x - l + \widetilde{\varepsilon})] - (u(x) - E[u(x + \widetilde{\varepsilon})]), \quad (3)$$

which is equivalent to:

$$w_P(x, \widetilde{\varepsilon}) = w_A(x - l, \tilde{\epsilon}) - w_A(x, \tilde{\epsilon}). \quad (4)$$

Naturally, $w_P(x, \widetilde{\varepsilon}) \geq 0$ if and only if $u''' \geq 0$[4].

Temperance is known as preference for disaggregation of two independent zero-mean risks. The temperance utility premium as introduced by Courbage and Rey (2010), denoted $w_T(x, \widetilde{\varepsilon})$, measures the increase in pain of facing the risk $\widetilde{\varepsilon}$ in the presence of an independent zero-mean risk $\tilde{\theta}$. It writes as follows:

$$w_T(x, \widetilde{\varepsilon}) = E[u(x + \tilde{\theta})] - E[u(x + \tilde{\theta} + \widetilde{\varepsilon})] - (u(x) - E[u(x + \widetilde{\varepsilon})]), \quad (5)$$

which is equivalent to:

$$w_T(x, \widetilde{\varepsilon}) = w_A(x + \tilde{\theta}, \tilde{\epsilon}) - w_A(x, \tilde{\epsilon}), \quad (6)$$

$w_T(x, \widetilde{\varepsilon}) \geq 0$ if and only if $u^{(4)} \leq 0$[5].

Courbage and Rey (2010) suggested an extension of these measures to higher orders defining the utility premium by iteration following Eeckhoudt and Schlesinger (2006). Denoting $w_{(2)}(x, \widetilde{\varepsilon})$ the Friedman and Savage (1948) utility premium of Eq. (2), we can proceed from their remark by defining for all n even and $n \geq 2$:

$$w_{(n+1)}(x, \tilde{\epsilon}) = w_{(n)}(x - l, \tilde{\epsilon}) - w_{(n)}(x, \tilde{\epsilon})$$

[3]We assume throughout this article that the utility function u is nth differentiable. As it is usual, we assume that the derivative of order k ($\forall k \geq 1$), denoted $u^{(k)}(x)$, has a constant sign in the domain of u: $u^{(k)}(x) \geq 0$ or $u^{(k)}(x) \leq 0 \; \forall x$. To simplify notations we will only write $u^{(k)} \geq 0$ or $u^{(k)} \leq 0$.

[4]To see this note that, using Eeckhoudt and Schlesinger (2006), Eq. (3) is positive by preference for disaggregation of harms by a prudent individual. Now, prudence defined in this way is equivalent to a positive third derivative of the utility function, by Jensen's inequality.

[5]Note that Eq. (5) is positive iff the individual is temperant by the expected utility equivalence of the definition of temperance in Eeckhoudt and Schlesinger (2006). Now, we see from Eq. (6) that $w_T \geq 0$ is equivalent to w_A is convex by Jensen's inequality. From the definition of w_A in Eq. (2) w_A is convex iff u'' is concave, which is equivalent to $u^{(4)} \leq 0$ by Jensen's inequality again.

with $l > 0$ and

$$w_{(n+2)}(x, \tilde{\epsilon}) = w_{(n)}(x + \tilde{\theta}_n, \tilde{\epsilon}) - w_{(n)}(x, \tilde{\epsilon}),$$

where $\tilde{\theta}_n$ is an independent random variable (i.e. random variables $\tilde{\epsilon}$, $\tilde{\theta}_2$, $\tilde{\theta}_4$, $\tilde{\theta}_6$, etc, are mutually independent) and such that $E(\tilde{\theta}_n) = 0$. As an illustration, when $n = 2$, $w_{(n+1)}(x, \tilde{\epsilon})$ corresponds to the prudence utility premium, $w_P(x, \tilde{\epsilon})$, and $w_{(n+2)}(x, \tilde{\epsilon})$ corresponds to the temperance utility premium, $w_T(x, \tilde{\epsilon})$, as defined by equations (4) and (6).

2.2. The RA-n utility premium

While Courbage and Rey (2010) suggested to define utility premia of higher orders by iteration of the previous premia of lower orders and in the context of specific lotteries, in this section we present a very general way to define the utility premium at higher orders using the concept of stochastic dominance of order n.

Let's consider two risky situations: a first situation represented by the random variable Y_1 and a second one represented by the random variable X_1. We assume that X_1 and Y_1 are independent, and that Y_1 dominates X_1 via nth-order stochastic dominance ($X_1 \preceq_{n-SD} Y_1$).

The concept of nth-order stochastic dominance is defined as follows[6]. Consider Y_1 and X_1 with F and G, respectively, their two cumulative distribution functions of wealth, defined over a probability support contained within the interval $[a, b]$. Define $F_1 = F$ and $G_1 = G$. Now define $F_{k+1}(z) = \int_a^z F_k(t)dt$ and $G_{k+1}(z) = \int_a^z G_k(t)dt$ for $k \geq 1$. The variable Y_1 dominates X_1 via nth-order stochastic dominance ($X_1 \preceq_{n-SD} Y_1$) if $F_n(z) \leq G_n(z)$ for all z, and if $F_k(b) \leq G_k(b)$ for $k = 1, 2, .., n$.

When the $n-1$ moments of X_1 and Y_1 are equals, nth-order stochastic dominance coincides with the Ekern's (1980) concept of increase in nth-order risk ($X_1 \preceq_n Y_1$). Ekern's (1980) definition includes the case of mean-preserving increase in risk of Rothshild and Stiglitz (1970) as well as of increase in downside risk defined by Menezes *et al.* (1980) as, respectively, a second-degree and a third-degree increase in risk.

Our objective is to define the non-monetary measure of the cost of facing the risk transition e.g. the passage from Y_1 to X_1 (with $X_1 \preceq_{n-SD} Y_1$). Let's define the function w as follows[7]:

$$w(x; Y_1, X_1) = E[u(x + Y_1)] - E[u(x + X_1)]. \tag{7}$$

The function $w(x; Y_1, X_1)$ measures the degree of pain associated with facing the passage from the risk Y_1 to the less favorable one, X_1, when the decision-maker's initial wealth is x. We formulate the following definition.

[6]See for example Jean (1980, 1984) or Ingersoll (1987).

[7]We assume throughout this article that the support of any random variable $\tilde{z}$ is defined such that $x + z$ is in the domain of u.

Definition. *Given two independent risks, Y_1 and X_1 such that Y_1 dominates X_1 via nth-order stochastic dominance ($X_1 \preceq_{n-SD} Y_1$), the function w defined as $w(x; Y_1, X_1) = E[u(x+Y_1)] - E[u(x+X_1)]$ is named the "risk apportionment of order n utility premium", also denoted the RA-n utility premium. It measures the degree of pain due to the aggravating nth-order stochastic dominance risk.*

From Ingersoll (1987), we observe that $w(x; Y_1, X_1) \geq 0$ if and only if $(-1)^{(1+k)} u^{(k)} \geq 0$ $\forall k = 1, \ldots, n$. Note that $(-1)^{(1+n)} u^{(n)} \geq 0 \; \forall n \geq 1$ means that all odd derivatives of u are positive and all even derivatives of u are negative. Following Brockett and Golden (1987) and according to Caballé and Pomansky (1996), an individual with such a utility function is said to be *mixed risk-averse*. Hence, for all order n, the RA-n utility premium of a mixed risk averse agent is always positive. In other words, such an individual always incurs a pain when facing the passage from the risk Y_1 to a less favorable one X_1 dominated via nth-order stochastic dominance. We will label a utility function u as mixed risk-averse of order n if it verifies $(-1)^{(1+k)} u^{(k)} \geq 0 \; \forall k = 1, \ldots, n$.

Particular cases of the RA-n utility premium are the various premia defined in the previous section for which Y_1 dominates X_1 via nth-order Ekern's dominance ($X_1 \preceq_n Y_1$). Indeed, following following Ekern (1980), if $X_1 \preceq_n Y_1$ then $w(x; Y_1, X_1) \geq 0$ if and only if $(-1)^{(1+n)} u^{(n)} \geq 0$. For instance, when $Y_1 = 0$ and X_1 is a zero-mean background risk, $X_1 = \tilde{\epsilon}$ with $E(\tilde{\epsilon}) = 0$, the function w writes as $w(x; 0, \tilde{\epsilon}) = u(x) - E[u(x+\tilde{\epsilon})] = w_A(x, \tilde{\varepsilon})$. It is the utility premium introduced by Friedman and Savage (1948) to define the non-monetary risk aversion measure: $w(x; 0, \tilde{\epsilon}) \geq 0$ if and only if the individual is risk-averse ($u'' \leq 0$). When Y_1 and X_1 are defined as equiprobable lotteries describing an increase in downside risk (Menezes et al. (1980)): $Y_1 = [-l, \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ and $X_1 = [0, -l + \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ with $l > 0$ and $E(\tilde{\epsilon}) = 0$, the function w writes as

$$w(x; Y_1, X_1) = \frac{1}{2} w_P(x, \tilde{\epsilon}), \tag{8}$$

where $w_P(x, \tilde{\epsilon})$ is the prudence utility premium defined by Crainich and Eeckhoudt (2008) which is positive if and only if $u^{(3)} \geq 0$. When Y_1 and X_1 are defined as the following lotteries: $Y_1 = [\tilde{\theta}, \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ and $X_1 = [0, \tilde{\theta} + \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$, with $\tilde{\theta}$ and $\tilde{\epsilon}$ independent and zero mean random variables ($E(\tilde{\theta}) = E(\tilde{\epsilon}) = 0$), the function w writes as

$$w(x; Y_1, X_1) = \frac{1}{2} w_T(x, \tilde{\epsilon}), \tag{9}$$

where $w_T(x, \tilde{\epsilon}) = E[u(x + \tilde{\theta})] + E[u(x + \tilde{\epsilon})] - u(x) - E[u(x + \tilde{\theta} + \tilde{\epsilon})]$ corresponds to the temperance utility premium defined by Courbage and Rey (2010) which is positive if and only if $u^{(4)} \leq 0$.

3. RA-n UTILITY PREMIUM AND CHANGES IN RISKS

As Courbage and Rey (2010), we can investigate, in the more general context of the RA-n utility premium, how this measure reacts to the introduction on wealth of a sure loss ($-l$ with $l > 0$) or a zero-mean background risk ($\tilde{\epsilon}$ with $E(\tilde{\epsilon}) = 0$). As intuition suggests, the pain increases in both cases under usual conditions on the signs of higher-orders derivatives of the utility function. Indeed, considering the impact of a sure loss and a background risk on the RA-n utility premium, we obtain:

$$w(x - l; Y_1, X_1) - w(x; Y_1, X_1) \geq 0 \Leftrightarrow (-1)^{(k+1)} u^{(k)} \geq 0 \ \ \forall k = 1, \ldots, n+1, \tag{10}$$

$$w(x + \tilde{\epsilon}; Y_1, X_1) - w(x; Y_1, X_1) \geq 0 \Leftrightarrow (-1)^{(k+1)} u^{(k)} \geq 0 \ \ \forall k = 1, \ldots, n+2. \tag{11}$$

These results are rather easy to obtain. Eq. (10) rewrites as $E[u(x+Y_1)] - E[u(x+X_1)] \leq E[u(x-l+Y_1)] - E[u(x-l+X_1)]$ which is equivalent to $E[u(x+Y_1)] + E[u(x-l+X_1)] \leq E[u(x+X_1)] + E[u(x-l+Y_1)]$. Using Theorem 3 of Eeckhoudt et al. (2009), we obtain that this inequality holds iff $(-1)^{(k+1)} u^{(k)} \geq 0 \ \forall k = 1, \ldots, n+1$. In the same vein, Eq. (11) rewrites as $E[u(x+Y_1)] - E[u(x+X_1)] \leq E[u(x+\tilde{\epsilon}+Y_1)] - E[u(x+\tilde{\epsilon}+X_1)]$ which is equivalent to $E[u(x+Y_1)] + E[u(x+\tilde{\epsilon}+X_1)] \leq E[u(x+X_1)] + E[u(x+\tilde{\epsilon}+Y_1)]$. Using Eeckhoudt et al. (2009), we obtain that this inequality holds iff $(-1)^{(k+1)} u^{(k)} \geq 0$ $\forall k = 1, \ldots, n+2$.

Eqs. (10) and (11) have intuitive explanations. If we consider a decision-maker with a mixed risk-averse utility function of order $n+2$, Eqs. (10) and (11) respectively mean that the RA-N utility premium ($\forall N \leq n+1$) is vulnerable to a sure loss, and that the RA-N utility premium ($\forall N \leq n$) is vulnerable to a zero-mean background risk.

Such analysis can be extended to a more general context by investigating the degree of pain associated with facing the passage from Y_1 to X_1 when the wealth level becomes riskier. A riskier wealth corresponds to the random wealth level, initially equal to $x + Y_2$, becoming $x + X_2$ where Y_2 dominates X_2 via sth-order stochastic dominance ($X_2 \preceq_{s-SD} Y_2$ with X_2 and Y_2 being independent random variables). The degree of pain associated with facing the passage from Y_1 to X_1 when the wealth level becomes riskier is defined by the following expression[8]:

$$w(x + X_2; Y_1, X_1) - w(x + Y_2; Y_1, X_1). \tag{12}$$

A positive sign of (12) means that the pain facing the passage from Y_1 to X_1 increases when the wealth level becomes riskier. We obtain the following proposition.

[8] Note that Eq. (10) (respectively (11)) corresponds to cases where $Y_2 = 0$ and $X_2 = -l$ with $l > 0$ (respectively $X_2 = \widetilde{\theta}$ with $E(\widetilde{\theta}) = 0$).

Proposition 1. *Consider mutually independent random variables X_1, Y_1, X_2, and Y_2, such that $X_1 \preceq_{n-SD} Y_1$ and $X_2 \preceq_{s-SD} Y_2$. Then $w(x+X_2;Y_1,X_1) - w(x+Y_2;Y_1,X_1) \geq 0$ for all utility functions u such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n+s$.*

Proof. Using the definition of w, $w(x+X_2;Y_1,X_1) - w(x+Y_2;Y_1,X_1) \geq 0$ rewrites equivalently as $E[u(x+Y_1+X_2)] - E[u(x+X_1+X_2)] \geq E[u(x+Y_1+Y_2)] - E[u(x+X_1+Y_2)]$, that is equivalent to $E[u(x+Y_1+X_2)] + E[u(x+X_1+Y_2)] \geq E[u(x+Y_1+Y_2)] + E[u(x+X_1+X_2)]$. Following Eeckhoudt et al. (2009), this last expression is equivalent to $(-1)^{(1+k)}u^{(k)} \geq 0$ for $k = 1, \ldots, n+s$. ■

Proposition 1 can be interpreted in a similar way as Eqs. (10) and (11). If we consider a decision-maker with a mixed risk-averse utility function of order $n+s$, Proposition 1 means that the RA-N utility premium ($\forall N \leq n-s$) is vulnerable to a detrimental change of order s in the background risk.

In order to provide a different interpretation of the results, we restrict our attention on the special case of Ekern dominance as often done in the literature. The advantage of this dominance rule is that the positive sign of the RA-n utility premium depends only on the sign of the nth derivative of u. Indeed, according to Ekern (1980), $w(x;Y_1,X_1) \geq 0$ if and only if $(-1)^{(1+n)}u^{(n)} \geq 0$, i.e. risk apportionment of order n holds. From Proposition 1, we can extrapolate the following corollary.

Corollary 1. *The four following items are equivalent:*

(1) risk apportionment of order n holds,

(2) the pain due to misaportionment of order $(n-1)$ is vulnerable to a sure loss,

(3) the pain due to misaportionment of order $(n-2)$ is vulnerable to a zero-mean background risk,

(4) the pain due to misaportionment of order $(n-s)$ is vulnerable to an increase in risk of order s.

Note that item (2) also means that the RA-*n-1* utility premium is decreasing in the wealth level x and item (3) means that the RA-*n-2* utility premium is convex in x. This is rather intuitive as when the individual gets richer the pain of facing increases in risks is reduced as he can deal with them more easily, but this reduction in pain diminishes as the individual's wealth increases.

Corollary 1 offers an alternative interpretation of the sign of the utility function n-order derivative ($u^{(n)}$) which can be easily understood and remembered, without reference to any specific decision problem. For instance in the case of the utility premium for which $Y_1 = 0$ and $X_1 = \tilde{\epsilon}$ ($\tilde{X}_1 \preceq_2 Y_1$), Corollary 1 tells us that the degree of pain of facing the risk $\tilde{\epsilon}$ increases when initial wealth becomes risky (i.e. when any zero-mean risk

X_2 is added to initial wealth) for a temperant individual. In the same way, in the case of the prudence premium, for which X_1 represents an increase in downside risk over Y_1 ($\tilde{X}_1 \preceq_3 Y_1$), Corollary 1 tells us that the degree of pain of facing an increase in downside risk increases when initial wealth becomes risky for an individual featuring edginess[9].

4. SUPERADDITIVITY OF THE RA-*n* UTILITY PREMIUM

An issue of importance when dealing with measures of risks is that of superadditivity or subadditivity of these measures. From the risk theory literature (see for example Buhlmann (1985) or Gerber and Goovaerts (1981)), it is well-known that financial risks are very often self-aggravating. This would suggest that the cost of risk for two independent risks should be greater than the sum of costs of the two risks taken in isolation. If it were the case, the cost of risk would be superadditive. Eeckhoudt and Gollier (2001) examine this issue when the cost of risk is defined in terms of risk premium. They show that the risk premium is superadditive if risk aversion is risk vulnerable[10].

We first address the issue of superadditivity when the cost of risk is defined in non-monetary terms through the concept of the Friedman-Savage utility premium. The definition of superadditivity is the following. A real-valued function f is superadditive if $f(n_1+n_2)$ is larger than $f(n_1)+f(n_2)$ for all $n_1 > 0$ and $n_2 > 0$. The opposite inequality holding true for subadditivity. We show that the Friedman-Savage utility premium is superadditive for all temperant decision-makers. This gives the following proposition.

Proposition 2. *Consider mutually independent zero-mean risks X_1 and X_2. Then the Friedman-Savage utility premium is superadditive (i.e. $w(x;0,X_1+X_2) \geq w(x;0,X_1) + w(x;0,X_2)$) for all utility functions u such that $u^{(4)} \leq 0$.*

Proof. Superadditivity of the Friedman-Savage utility premium rewrites as $u(x) - E[u(x+X_1+X_2)] \geq u(x) - E[u(x+X_1)] + u(x) - E[u(x+X_2)]$. Using Eeckhoudt et al. (2009), this inequality is equivalent to $w_T \geq 0$, i.e. $u^{(4)} \leq 0$ since the introduction of a zero-mean risk X_i $(i = 1,2)$ is an increase in risk of order 2 in the sense of Ekern. ■

Proposition 2 states that the pain of facing two risks simultaneously is higher than the sum of the pains of facing each risk separately for a temperant individual. Let's make two remarks. First, while risk vulnerability is required for monetary measures of risk to be superadditive, for which temperance is only a necessary condition, temperance here is

[9] The concept of edginess, i.e. $u^{(5)} \geq 0$, was introduced by Lajeri-Chaherli (2004) to explain the effects of background risks on precautionary savings.

[10] Risk vulnerability means that risk aversion increases with the presence of an independent background risk (Gollier and Pratt, 1996). Sufficient and necessary conditions on the utility function to have risk vulnerability are quite complex. A necessary condition for risk vulnerability is $u^{(4)} \leq 0$.

sufficient to obtain superadditivity in the case of the Fiedman-Savage utility premium. Second, obviously the superadditivity of the Friedman-Savage utility premium holds for all "mixed risk-averse of order 4" decision-makers.

Extending superadditivity to the RA-n utility premium allows us to consider the more following general context. Instead of considering the passage from 0 to X_i ($i = 1, 2$), we consider the passage from a risky situation Y to a less favorable one X_i ($i = 1, 2$). The superadditivity to the RA-n utility premium writes as

$$w(x; Y, X_1 + X_2) \geq w(x; Y, X_1) + w(x; Y, X_2), \tag{13}$$

where Y, X_1 and X_2 are zero-mean mutually independent random variables. Subadditivity of the utility premium corresponds naturally to the opposite inequalities. We obtain the following result.

Proposition 3. *Consider mutually independent zero-mean risks, X_1, X_2 and Y. Then the RA-n utility premium is superadditive (i.e. $w(x; Y, X_1 + X_2) \geq w(x; Y, X_1) + w(x; Y, X_2)$) for all utility functions u such that $u'' \leq 0$ and $u^{(4)} \leq 0$.*

Proof. $w(x; Y, X_1 + X_2) \geq w(x; Y, X_1) + w(x; Y, X_2)$ is equivalent to $E[u(x + X_1)] + E[u(x + X_2)] \geq E[u(x + X_1 + X_2)] + E[u(x + Y)]$. If $u'' \leq 0$ then $E[u(x + Y)] \leq u(x)$ and then $E[u(x + X_1 + X_2)] + E[u(x + Y)] \leq E[u(x + X_1 + X_2)] + u(x)$. If $u^{(4)} \leq 0$ then $E[u(x + X_1 + X_2)] + u(x) \leq E[u(x + X_1)] + E[u(x + X_2)]$ (Eeckhoudt et al. (2009)). Consequently, if $u'' \leq 0$ and $u^{(4)} \leq 0$ then $E[u(x + X_1 + X_2)] + E[u(x + Y)] \leq E[u(x + X_1)] + E[u(x + X_2)]$. ■

According to Proposition 3, the pain of facing a change in two risks simultaneously is higher than the sum of the pains of facing two changes in risk separately for all risk-averse and temperant decision-makers. Obviously the result holds for all "mixed risk-averse of order 4" decision-makers. Note that Proposition 2 holds for all temperant decision makers i.e. for all u such that $u^{(4)} \leq 0$ whatever the sign of other lower order derivatives contrary to Proposition 3 that imposes risk aversion ($u'' \leq 0$). Risk aversion is not required in the case of the Savage-Friedman utility premium because assuming $Y = 0$ implies that superadditivity rewrites equivalently as preference for harms disaggregation (Eeckhoudt et al. (2009)). In Proposition 3 (where Y is a zero-mean independent risk) it is no longer the case. Imposing risk aversion ensures that $E[u(x + Y)] \leq u(x)$, and then coupled with $u^{(4)} \leq 0$ is sufficient to obtain superadditivity. In the case of a risk lover, $E[u(x+Y)] \geq u(x)$ and then the sum $E[u(x+X_1+X_2)]+E[u(x+Y)]$ can be larger than the sum $E[u(x+X_1)]+E[u(x+X_2)]$. Note that Proposition 3 holds whatever the nature of the change in risk between Y and respectively X_1 and X_2. Indeed, for any stochastic dominance levels n and s such that $X_1 \preceq_{n-SD} Y$ and $X_2 \preceq_{s-SD} Y$, Proposition 3 holds.

5. MERGING CHANGES IN RISKS

While the previous section analyzes superadditivity of the RA-n utility premium, this section addresses merging properties.

To do so, we consider a change in risk as the passage from Y_1 to X_1 and from Y_2 to X_2 with $X_1 \preceq_{n-SD} Y_1$ and $X_2 \preceq_{s-SD} Y_2$ where risks Y_1, Y_2, X_1 and X_2 are mutually independent and where Y_1 and Y_2 are zero-mean risks. We wonder under which conditions on the utility function u the non monetary cost of the total change in risk (passage from (Y_1+Y_2) to (X_1+X_2)) is larger than the sum of the non-monetary cost of each change in risk (passage from Y_1 to X_1 and simultaneously passage from Y_2 to X_2). More formally, we wonder what properties of u ensure the following inequality:

$$w(x; Y_1+Y_2, X_1+X_2) \geq w(x; Y_1, X_1) + w(x; Y_2, X_2). \tag{14}$$

The following proposition provides conditions for such a comparison.

Proposition 4. *Consider mutually independent random variables X_1, X_2, Y_1 and Y_2, such that $X_1 \preceq_{n-SD} Y_1$, $X_2 \preceq_{s-SD} Y_2$, with $E(Y_1) = E(Y_2) = 0$. Then the RA-n utility premium verifies $w(x; Y_1+Y_2, X_1+X_2) \geq w(x; Y_1, X_1) + w(x; Y_2, X_2)$ for all utility functions u such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n+s$, $\forall n \geq 2$, $\forall s \geq 2$.*

Proof. Since $E(Y_1) = 0$ we have $Y_1 \preceq_2 0$. Given that $X_2 \preceq_s Y_2$ and applying Eeckhoudt et al. (2009) theorem, we know that $E[u(x+Y_2)] + E[u(x+X_2+Y_1)] - E[u(x+X_2)] - E[u(x+Y_1+Y_2)] \leq 0$ for all u such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \ldots, s+2$.

Analogously, since $E(Y_2) = 0$ we have $Y_2 \preceq_2 0$. Given that $X_1 \preceq_n Y_1$ and applying Eeckhoudt et al. (2009) theorem, we know that $E[u(x+Y_1)] + E[u(x+X_1+Y_2)] - E[u(x+X_1)] - E[u(x+Y_1+Y_2)] \leq 0$ for all u such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n+2$. However, Eeckhoudt et al. (2009) theorem gives $E[u(x+Y_1+Y_2)] + E[u(x+X_1+X_2)] - E[u(x+X_1+Y_2)] - E[u(x+Y_1+X_2)] \leq 0$ for all u such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n+s$. Consequently, if u is such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n+s$, $\forall n \geq 2$, $\forall s \geq 2$ then the following inequality holds: $\Big(E[u(x+Y_2)] + E[u(x+X_2+Y_1)] - E[u(x+X_2)] - E[u(x+Y_1+Y_2)]\Big) + \Big(E[u(x+Y_1)] + E[u(x+X_1+Y_2)] - E[u(x+X_1)] - E[u(x+Y_1+Y_2)]\Big) + \Big(E[u(x+Y_1+Y_2)] + E[u(x+X_1+X_2)] - E[u(x+X_1+Y_2)] - E[u(x+Y_1+X_2)]\Big) \leq 0$. It rewrites equivalently as: $E[u(x+Y_1)] - E[u(x+X_1)] + E[u(x+Y_2)] - E[u(x+X_2)] \leq E[u(x+Y_1+Y_2)] - E[u(x+X_1+X_2)$ that is equivalent to $w(x; Y_1, X_1) + w(x; Y_2, X_2) \leq w(x; Y_1+Y_2, X_1+X_2)$ that ends the proof. ■

Proposition 4 refers to a merging property. It means that welfare is reduced by merging changes in risks instead of facing them separately, i.e. the welfare loss of both increases in risks taken together is larger than the sum of welfare losses from assuming each increase in risk separately.

Note that when $Y_1 = Y_2 = 0$ (that implies considering $n = s = 2$) and $E(X_1) = E(X_2) = 0$, the superadditivity property of the Friedman-Savage utility premium coincides with this merging property. Indeed inequality (14) rewrites as

$$w(x; 0, X_1 + X_2) \geq w(x; 0, X_1) + w(x; 0, X_2)$$

which is precisely the inequality describing the superadditivity of the Friedman-Savage utility premium in Proposition 2. Hence, superadditivity of the Friedman-Savage utility premium means that welfare is reduced by merging increases in risks instead of facing them separately. This result is reminiscent of Samuelson (1963) who points out that risk-averters prefer to subdivide risks instead of facing them in one shot. Similarly, Eeckhoudt et al. (1991) show that risk-averters reduce their demand for insurance with a positive loading when the total property at risk is scattered on smaller independent units.

In the context described by Proposition 4, the two risk sets (Y_1, X_1) and (Y_2, X_2) are considered in isolation. In Eq. (14), on the RHS, there is no link between (Y_1, X_1) on one hand and (Y_2, X_2) on the other hand. Nevertheless, these two risk sets are present in the DM's environment. A change of perspective arises if we take this fact into account, and if we consider that risk 2 is a background risk for the management of risk 1, and inversely. Interestingly, it turns out that this change of perspective reverses the result. For example, in the case of the Friedman-Savage utility premium, we obtain the following proposition.

Proposition 5. *Given mutually independent zero-mean random variables, X_1 and X_2, the Friedman-Savage utility premium verifies $w(x; 0, X_1 + X_2) \leq w(x + X_2; 0, X_1) + w(x + X_1; 0, X_2)$ for all utility functions u such that $u^{(4)} \leq 0$.*

Proof. The inequality $w(x; 0, X_1 + X_2) \leq w(x + X_2; 0, X_1) + w(x + X_1; 0, X_2)$ rewrites as $\Big(E[u(x + X_2)] - E[u(x + X_2 + X_1)]\Big) + \Big(E[u(x + X_1)] - E[u(x + X_1 + X_2)]\Big) \geq u(x) - E[u(x + X_1 + X_2)]$, equivalent to $E[u(x + X_2)] + E[u(x + X_1)] \geq u(x) + E[u(x + X_1 + X_2)]$. Following Eeckhoudt et al. (2009) theorem, this inequality holds for all utility function u verifying $u^{(4)} \leq 0$. ■

Proposition 5 means that the welfare loss of facing the total increase in risk (passage from 0 to $(X_1 + X_2)$) is smaller than the welfare loss of facing the first increase in risk (passage from 0 to X_1) when the second risk X_2 is considered as a background risk (i.e. a risk that impacts the wealth level x) plus the welfare loss of facing the second increase in risk (passage from 0 to X_2) when the first X_1 is considered as a background risk. The result can also be generalized, yielding the following proposition.

Proposition 6. *Consider mutually independent random variables X_1, X_2, Y_1, and Y_2 such that $X_1 \preceq_{n-SD} Y_1$ and $X_2 \preceq_{s-SD} Y_2$. Then the two following items (a) and (b) hold for all utility functions u such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \dots, n+s$:*
(a) $w(x; Y_1 + Y_2, X_1 + X_2) \leq w(x + X_2; Y_1, X_1) + w(x + X_1; Y_2, X_2)$,
(b) $w(x; Y_1 + Y_2, X_1 + X_2) \geq w(x + Y_2; Y_1, X_1) + w(x + Y_1; Y_2, X_2)$.

Proof. Item (a) rewrites as: $E[u(x+Y_1+Y_2)] - E[u(x+X_1+X_2)] \leq \Big(E[u(x+X_2+Y_1)] - E[u(x+X_2+X_1)]\Big) + \Big(E[u(x+X_1+Y_2)] - E[u(x+X_1+X_2)]\Big)$ that is equivalent to $E[u(x+Y_1+Y_2)]+E[u(x+X_1+X_2)] \leq E[u(x+Y_2+X_1)]+E[u(x+Y_1+X_2)]$. Following Eeckhoudt et al. (2009) theorem, this inequality holds for all u such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \dots, n+s$. Item (b) rewrites as: $E[u(x + Y_1 + Y_2)] - E[u(x + X_1 + X_2)] \geq \Big(E[u(x+Y_2+Y_1)] - E[u(x+Y_2+X_1)]\Big) + \Big(E[u(x+Y_1+Y_2)] - E[u(x+Y_1+X_2)]\Big)$ that is equivalent to $E[u(x+Y_1+Y_2)]+E[u(x+X_1+X_2)] \leq E[u(x+Y_2+X_1)]+E[u(x+Y_1+X_2)]$ i.e. the same inequality as the one of item (a). ■

Item (a) is the generalization of Proposition 5 (where $Y_1 = Y_2 = 0$, $n = s = 2$ and $E(X_1) = E(X_2) = 0$). But in this general case, background risks can also be Y_1 and Y_2 that provides item (b). Note also that in the case where $Y_1 = Y_2 = 0$, item (b) rewrites as $w(x; 0, X_1 + X_2) \geq w(x; 0, X_1) + w(x; 0, X_2)$ that is the result of Proposition 2.

The intuitive interpretation of Proposition 6 is the following. The difference between items (a) and (b) arises from background risk considerations on the RHS. In item (a), the background risks taken into account are the worse risks X_1 and X_2 while they are the better ones Y_1 and Y_2 in item (b). In item (a), the decision-maker is aware that risk Y_2 will be replaced by risk X_2 when she feels the loss of welfare from facing the risk X_1 instead of Y_1. In item (b), the decision-maker is blind to this risk substitution. She feels a reduced loss from the sum of individual risk substitutions because she ignores that risk Y_2 will be replaced by risk X_2 when dealing with risk 1, and she ignores that risk Y_1 will be replaced by risk X_1 when dealing with risk 2. In this sense, item (a) reflects a kind of rational expectations, whereas item (b) reflects blindness.

It should also be stressed that following Ebert et al. (2017), item (b) of Proposition 6 can have another interpretation in terms of mutual aggravation of risks changes. Using the perspective of these authors, we show that item (a) can also be interpreted in terms of mutual improvement of risks changes.

To reach such conclusion, we have to consider the following property of the RA-n utility premium that stands from the fact that risks are additive:

Property 1. *Given mutually independent random variables, X_1, Y_1 and Z. Then*

$w(x+Z; Y_1, X_1) = w(x; Y_1+Z, X_1+Z)$.

The proof is straightforward. By definition of the RA-n utility premium, we have: $w(x+Z; Y_1, X_1) = E[u(x+Z+Y_1)] - E[u(x+Z+X_1)] = w(x; Y_1+Z, X_1+Z)$. Using this property, Proposition 6 can be rewritten as follows.

Corollary 2. *Consider mutually independent random variables X_1, X_2, Y_1, and Y_2 such that $X_1 \preceq_{n-SD} Y_1$ and $X_2 \preceq_{s-SD} Y_2$. Then the two following items (a) and (b) hold for all utility functions u such that $(-1)^{(1+k)}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n+s$:*
(a) $w(x; Y_1+Y_2, X_1+X_2) \leq w(x; X_2+Y_1, X_2+X_1) + w(x; X_1+Y_2, X_1+X_2)$,
(b) $w(x; Y_1+Y_2, X_1+X_2) \geq w(x; Y_2+Y_1, Y_2+X_1) + w(x; Y_1+Y_2, Y_1+X_2)$.

Item (b) of Corollary 2 is identical to Eq. (1) of Ebert et al. (2017)[11], which according to these authors means that the utility from avoiding both risk changes at once is larger than the utility from avoiding the first risk change plus the utility from avoiding the second risk change. In other words, Ebert et al. (2017) reinterpret preferences for combining *good* with *bad* given by Eeckhoudt et al. (2009) to explain the alternance of signs of successive derivatives as *mutual aggravation of risk changes,* i.e. the decision-maker's trait of perceiving two risks changes as mutually aggravating.

Following Ebert et al. (2017), we can provide a new interpretation of item (a) of Corollary 2 in terms of mutual improvement as follows: the increase in utility due to the two improvements at once (passage from (X_1+X_2) to (Y_1+Y_2)) is smaller than the increase in utility due to the improvement of the first risk plus the increase in utility due to the improvement of the second risk. This provides a new interpretation of preferences for combining *good* with *bad* as *mutual improvement of risk changes,* i.e. the decision-maker's trait of perceiving two risks changes as mutually improving.

6. CONCLUSION

The paper provides a generalization of non-monetary measures of risk by introducing the concept of risk apportionment of order n (RA-n) utility premium. This measure reflects the degree of pain due to facing the transition from one risk to a more severe one. Changes in risks are expressed through the concept of stochastic dominance of order n. Risk apportionment is taken as a starting point using the definitions of attitudes towards risk introduced by Eeckhoudt and Schlesinger (2006). The prudence utility premium and the temperance utility premium are special cases of our RA-n utility premium.

[11] This inequality is equivalent to: $E[u(x+Y_1+Y_2)] + E[u(x+X_1+X_2)] \leq E[u(x+Y_1+X_2)] + E[u(x+Y_2+X_1)]$ given by Eeckhoudt et al. (2009) to interpret the preference for combining *good* with *bad*.

We first show that the RA-n utility premium increases when the decision-maker faces a riskier wealth under mixed risk aversion. We then turn to the issue of deciding whether the RA-n utility premium is subadditive or superadditive, i.e., deciding whether the cost of an increase in several risks faced jointly is smaller or larger than the sum of these increases in risks faced independently. We obtain that the Friedman-Savage utility premium for an individual with no initial risk is superadditive if the fourth derivative of the utility function is negative (the decision-maker is temperant). We further show that the more general RA-n utility premium is superadditive if the decision-maker is mixed risk averse of order 4. We finally address a related but different issue which is whether it is valuable to merge risks instead of facing them in separate entities. Our results show that an individual whose preferences are mixed-risk averse will feel more pain from merging increases in risk than from facing them in separate entities. However, the result is reversed if risks in separate entities are considered as background risks of each other, and if the individual is aware that both risks will deteriorate. Links with the concept of mutual aggravation of risks introduced by Ebert et al. (2017) are also provided.

The results in this paper provide new interpretations of the sign of higher derivatives of the utility function. As all commonly used utility functions in economic theory, with the first derivative being positive and the second one being negative, exhibit mixed risk aversion, our results then apply to most individuals facing a deterioration in their risk environment.

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